SHARP LIPSCHITZ CONSTANTS FOR THE DISTANCE RATIO METRIC

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Conformal invariants and conformally invariant metrics have been some of the key notions of geometric function theory and of quasiconformal mapping theory for several decades [AVV]. One of the modern trends is to extend this theory to Euclidean spaces of higher dimension or to more general metric spaces.

We study here expansion/contraction properties of Möbius transformations of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ onto itself with respect to the distance ratio metric.

For a subdomain $G \subset \mathbb{R}^n$ and for all $x, y \in G$ the distance-ratio metric j_G is defined as

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right),$$

where $d(x,\partial G)$ denotes the Euclidean distance from the point x to the boundary ∂G of the domain G. The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [GP].

The distance ratio metric j_G is not invariant under Möbius transformation. Therefore, it is natural to ask what the Lipschitz constants are for this metric under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring and B. G. Osgood proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [GO, proof of Theorem 4].

Theorem 0.1. *If D and D' are proper subdomains of $\mathbb{R}^n$ and if f is a Möbius transformation of D onto D', then for all $x, y \in D$*

$$\frac{1}{2} j_D(x,y) \le j_{D'}(f(x), f(y)) \le 2 j_D(x,y)$$

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This global estimation can be improved for some special domains.

For example, an answer to the conjecture proposed in [KVZ] is given in the following assertion.

Theorem 0.2. *Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ be a Möbius mapping with $f(a) = 0$. Then for all $x, y \in \mathbb{B}^n$*

$$\frac{1}{1+|a|} j_{\mathbb{B}^n}(x,y) \le j_{\mathbb{B}^n}(f(x), f(y)) \le (1+|a|) j_{\mathbb{B}^n}(x,y),$$

and the constants $\frac{1}{1+|a|}$ and $1+|a|$ are both best possible.

Remark 0.3. *The sharpness of the constant is proved in* [KVZ, Remark 3.4] *by taking $x = ta/|a| = -y$, $t \in (0,1)$ and letting $t \to 0^+$.*

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For a punctured disk we obtain

Theorem 0.4. *Let $a \in \mathbb{B}^2$ and $f : \mathbb{B}^2 \setminus \{0\} \to \mathbb{B}^2 \setminus \{a\}$ be a Möbius transformation with $f(0) = a$. Then for x, $y \in \mathbb{B}^2 \setminus \{0\}$*

$$j_{\mathbb{B}^2 \setminus \{a\}}(f(x), f(y)) \leq C(a) j_{\mathbb{B}^2 \setminus \{0\}}(x, y),$$

where the constant $C(a) = 1 + (\log \frac{2+|a|}{2-|a|})/\log 3$ is best possible and $C(a) < 1 + |a|$.

References

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