Absolute Arithmetic and $\mathbb{F}_1$-geometry

http://cage.ugent.be/ kthas/Fun/index.php/absolute-arithmetic-at-the-6th-european-congress-of-mathematics.html

ORGANIZER: Koen Thas (*Ghent University, BE)*

Wednesday, July 4, 14:30–16:30, Small Hall

TALKS:
Koen Thas (*Ghent University, BE*), **Foundations of absolute arithmetic (Introductory talk)**

Yuri I. Manin (*Max Planck Institute, Bonn, DE and Northwestern University, Evanston, USA*), **F1-analytic functions and Borger's descent**

Oliver Lorscheid (*IMPA, Rio de Janeiro, BR*), **F1-geometry and its applications**

Lieven Le Bruyn (*University of Antwerp, BE*), **Noncommutative geometry and F1**

Foundations of absolute arithmetic (Introductory talk)

Koen Thas
Ghent University, BE

In a paper which was published in 1957, Tits made a seminal and provocative remark which alluded to the fact that through a certain analogy between the groups $\mathbf{GL}_n(q)$ | q a prime power | and the symmetric group $\mathbf{S}_n$, one should interpret the latter as a Chevalley group "over the field of characteristic one", $\mathbb{F}_1$.

In this introductory lecure, I will survey some of Tits's ideas from a combinatorial point of view, as well as mention the linear algebra in characteristic one which was developed by Kapranov and Smirnov.

Then I will describe Anton Deitmar's geometry of commutative monoids, which is a fundamental attempt to introduce an algebraic geometry over $\mathbb{F}_1$ which lives deeply "below $\mathrm{Spec}(\mathbb{Z})$".

F1-analytic functions and Borger's descent

Yuri I. Manin
Max Planck Institute, Bonn, DE and Northwestern University, Evanston, USA

The existence of algebraic geometry over $\mathbb{F}_1$, a nonexistent "field with one element", was tentatively suggested by Jacques Tits in 1957. In the last ten years or so the interest of mathematicians in this idea was steadily growing, and by now it has developed into an active (if small) research field. Not one but about a dozen versions of $\mathbb{F}_1$-geometry appeared. In this talk I will survey two recent ideas that emerged and were tested in these studies:

1. Analytic geometry over $\mathbb{F}_1$, suggested by the speaker and based on the notion of multivariable Habiro rings, that (in the one variable version) was initially introduced in totally different contexts as generating series for certain knot and other topological invariants.
2. Borger's geometry, based on the idea that the "descent data" on a commutative ring coming from an affine $\mathbb{F}_1$-scheme are given by the lambda-structure upon this ring. In particular, I will discuss Borger's descent for Habiro rings and its relation to Witt rings.

F1-geometry and its applications

Oliver Lorscheid
IMPA, Rio de Janeiro, BR

The original motivation to consider a geometry over the *field* $\mathbb{F}_1$ *with one element* is based on a remark by Jacques Tits in a paper from 1957: it seemed that analogies between geometry over finite fields and combinatorial geometry could find an explanation in such a theory. In the early nineties, Manin, Smirnov, et al., connected this viewpoint to arithmetic problems like the abc-conjecture and the Riemann hypothesis. From then on, the field with one element attracted more and more attention.

In the last decade, there appeared more than a dozen different definitions of a geometry over $\mathbb{F}_1$. Though a solution of the mentioned arithmetic problems is out of sight yet, it became clear that $\mathbb{F}_1$-geometry connects to many other fields of mathematics and that it is the natural formulation to understand problems with a combinatorial flavour in an algebro-geometric language. This touches stable homotopy of spheres; tropical geometry and log-schemes; non-archimedean analytic spaces and Arakelov theory; reductive groups and buildings; canonical bases and cluster algebras; moduli of quiver representations and quiver Grassmannians.

In this talk, I will explain some of the ideas of $\mathbb{F}_1$-geometry and review its achievements so far.

Noncommutative geometry and F1

Lieven Le Bruyn
University of Antwerp, BE

There is already a touch of noncommutativity present in most approaches to commutative algebraic geometry over $\mathbb{F}_1$ via natural actions of the Bost-Connes algebra. This is a canonical object associated to the group of all roots of unity and the action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on them, two main ingredients in most approaches to $\mathbb{F}_1$.

We will explore generalizations of Soulé's and other approaches to $\mathbb{F}_1$-geometry to the truly noncommutative realm. We will give an example in which the role of the roots of unity and their Galois action is replaced by that of Grothendieck's "dessins d'enfants" and the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

We will use techniques from noncommutative algebraic geometry, in particular work by Kontsevich and Soibelman, to define and calculate substitutes of Habiro's new topology on roots of unity and its associated functions in this wilder noncommutative setting.