Cutting the same fraction of several measures

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Abstract

The famous "ham sandwich" theorem of Stone, Tukey, and Steinhaus asserts that every d absolutely continuous probability measures in $\mathbb{R}^d$ can be simultaneously partitioned into equal parts by a single hyperplane.

We consider the following question: If we are given $d+1$ measures in $\mathbb{R}^d$ and want to cut the same (but unknown) fraction of every measure by a hyperplane then what assumptions on the measures allow us to do so? We call the set μ_0, μ_1, ..., μ_d of measures ε-*not-permuted* if for any halfspace H the inequalities $\mu_i(H) < \varepsilon$ for all $i = 0, 1, \dots, d$ imply $\mu_i(H) \geq \mu_j(H)$, for some $i < j$.

Theorem: *Suppose μ_0, μ_1, ..., μ_d are absolutely continuous probability ε-not-permuted measures in $\mathbb{R}^d$ for some $\varepsilon \in (0, 1/2)$. Then there exists a halfspace H such that $\mu_0(H) = \mu_1(H) = \cdots = \mu_d(H) \in [\varepsilon, 1/2]$.*

We also consider a problem of cutting the same *prescribed* fraction of every measure, allowing cutting with a convex subset of $\mathbb{R}^d$.

Theorem: *Suppose μ_0, μ_1, ..., μ_d are absolutely continuous probability measures on $\mathbb{R}^d$ and $\alpha \in (0, 1)$. It is always possible to find a convex subset $C \subset \mathbb{R}^d$ such that $\mu_0(C) = \mu_1(C) = \cdots = \mu_d(C) = \alpha$, if and only if $\alpha = 1/m$ for a positive integer m.*

AMS Classification: 52-XX.