Bernoulli numbers, Drinfeld associators, and the Kashiwara–Vergne problem

(based on joint works with B. Enriquez, E. Meinrenken, M. Podkopaeva, P. Severa, C. Torossian)

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Bernoulli numbers

Jacob Bernoulli

$$1^m + 2^m + \cdots + n^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}$$

$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ \ B_{2k+1} = 0$, for $k \geq 1$

Generating function:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$$

$$\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{k=2}^{\infty} B_k \frac{t^k}{k!}$$

Campbell–Hausdorff series

$x, y, z =$ generators of a free Lie algebra

$$\mathrm{ch}(x,y) = \log(e^x e^y) = x + \frac{\mathrm{ad}_x}{1 - e^{-\mathrm{ad}_x}} y + O(y^2)$$

$$\left(\frac{\mathrm{ad}_x}{1 - e^{-\mathrm{ad}_x}} y = y + \frac{1}{2}[x,y] + \sum_{k=2}^{\infty} \frac{B_k}{k!} \mathrm{ad}_x^k (y) \right)$$

Theorem

$\mathrm{ch}(x,y)$ *is the unique Lie series such that*

- $\mathrm{ch}(x,y) = x + y + \frac{1}{2}[x,y] + \dots$
- $\mathrm{ch}(x, \mathrm{ch}(y,z)) = \mathrm{ch}(\mathrm{ch}(x,y), z)$

Duflo isomorphism

$\mathbb{K}$ = field of characteristic 0,

$\mathfrak{g}$ = Lie algebra over $\mathbb{K}$, $\dim \mathfrak{g} < +\infty$

Theorem (Duflo, 1977)

$$Z(U\mathfrak{g}) \cong (S\mathfrak{g})^{\mathfrak{g}}$$

$Z(U\mathfrak{g})$ = the center of the universal enveloping algebra

$(S\mathfrak{g})^{\mathfrak{g}}$ = the ring of invariant polynomials

Notation

$$V = \text{vector space} \qquad SV = \mathbb{K}[x_1, \dots, x_n]$$
$$V^* = \text{its dual} \qquad \overline{SV^*} = \mathbb{K}[[p_1, \dots, p_n]]$$

Consider elements of $\overline{SV^*}$ as (possibly) infinite order constant coefficient differential operators

$$p_i \mapsto \frac{\partial}{\partial x_i}$$

Example: $n = 1$

$$\begin{aligned} A &= a_0 + a_1 p + \dots + a_k p^k + \dots \\ &\mapsto \partial_A = a_0 + a_1 \frac{\mathrm{d}}{\mathrm{d}x} + \dots + a_k \frac{\mathrm{d}^k}{\mathrm{d}x^k} + \dots \end{aligned}$$

Duflo isomorphism

The isomorphism $(S\mathfrak{g})^{\mathfrak{g}} \cong Z(U\mathfrak{g})$ is a restriction (to $\mathfrak{g}$ invariants) of the vector space isomorphism

$$\mathrm{Duf} = Sym \circ \partial_{J^{1/2}}$$

where $Sym : S\mathfrak{g} \to U\mathfrak{g}$ is the symmetrization map: $xy \mapsto \frac{1}{2}(xy + yx)$

and

$$\begin{aligned} J^{\frac{1}{2}}(x) &= \left(\det\left(\frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}\right)\right)^{\frac{1}{2}} = \\ &= exp\left(\frac{1}{2}\mathrm{Tr}\,\mathrm{ad}_x + \frac{1}{2}\sum_{k=2}^{\infty}\frac{B_k}{k\cdot k!}\mathrm{Tr}(\mathrm{ad}_x^k)\right) \in \overline{S\mathfrak{g}^*} \end{aligned}$$

Example

- $\mathfrak{g} = su(2) = \langle x, y, z \rangle$
  $[x, y] = z, \quad [y, z] = x, \quad [z, x] = y$
- $\mathfrak{g}^* = \mathbb{R}^3, \quad (S\mathfrak{g})^{\mathfrak{g}} = \mathbb{R}[x^2 + y^2 + z^2]$
- the Casimir element $x^2 + y^2 + z^2 \in Z(U\mathfrak{g})$

$$\mathrm{Duf} : x^2 + y^2 + z^2 \mapsto x^2 + y^2 + z^2 + \frac{1}{4}$$

$$\partial_{J^{\frac{1}{2}}} = 1 + \frac{1}{24}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}) + \dots$$

Questions

- How difficult is the Duflo theorem?

  Proofs:

  Duflo (1977) $\Leftarrow$ structure theory
  Kontsevich (1997) $\Leftarrow$ graphical calculus
  Torossian, A.A. (2008) $\Leftarrow$ Drinfeld associators

- Why Bernoulli numbers?

Kashiwara–Vergne conjecture

1978

$\exists$ $A(x,y)$, $B(x,y)$ Lie series in x and y, such that

1. $x + y - \log(e^y e^x) = (1 - e^{-\mathrm{ad}_x})A + (e^{\mathrm{ad}_y} - 1)B$,
2. $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_x \circ \partial_x A + \mathrm{ad}_y \circ \partial_y B) = \frac{1}{2}\mathrm{tr}_{\mathfrak{g}}\left(\frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x}-1} + \frac{\mathrm{ad}_y}{e^{\mathrm{ad}_y}-1} - \frac{\mathrm{ad}_z}{e^{\mathrm{ad}_z}-1} - 1\right)$.

Notation:
- $z = \log(e^x e^y)$,
- $\partial_x A : \mathfrak{g} \to \mathfrak{g}$, $\partial_x A(u) = \frac{\mathrm{d}}{\mathrm{d}t} A(x + tu, y)|_{t=0}$

KV conjecture

Remark: $\dim \mathfrak{g} < +\infty \implies \mathrm{tr}_{\mathfrak{g}}$ well-defined

Theorem (Kashiwara, Vergne)

KV conjecture $\implies$ *Duflo isomorphism*

Remark: Equation (1) is easy to solve

$$a = \frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x} A \qquad b = \frac{e^{\mathrm{ad}_y} - 1}{\mathrm{ad}_y} B$$

$$x + y - \log(e^y e^x) = [x, a] + [y, b]$$

$\implies$ many rational solutions

Definition: $\mathfrak{g}$ is a quadratic Lie algebra if it carries a non-degenerate symmetric bilinear form:

$$Q : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K},$$

$$Q([x, y], z) + Q(y, [x, z]) = 0.$$

Example: $\mathfrak{g}$ semisimple, Q Killing form
+ many other examples

Theorem (Torossian, A.A.)

For $\mathfrak{g}$ quadratic,

$$KV1 \implies KV2$$

Theorem

The KV conjecture holds true for all finite-dimensional Lie algebras.

Meinrenken, A.A	2006, using Kontsevich graphical calculus
Torossian, A.A	2008, using Drinfeld associators

Drinfeld associators

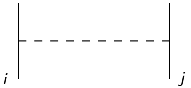
Lie algebra of infinitesimal pure braids $\mathfrak{t}_n$

Generators: $t_{i,j},\ i,j = 1,\ldots,n$

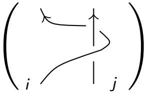
Relations:

$$t_{i,i} = 0, \quad t_{i,j} = t_{j,i},$$
$$[t_{i,j}, t_{k,l}] = 0, \quad i,j,k,l \text{ distinct}$$
$$[t_{i,j} + t_{i,k}, t_{j,k}] = 0.$$

In knot theory:



$= \quad t_{i,j} \quad \approx \quad \log$



Drinfeld associators

Notation: $t_{i,jk} = t_{i,j} + t_{i,k}$

Definition

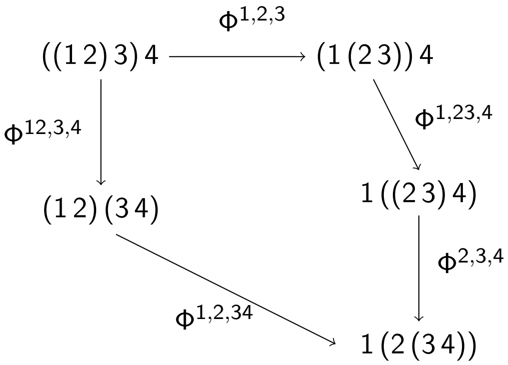
$\Phi \in \mathbb{K} \ll x, y \gg$ is a Drinfeld associator if

1. Φ is group-like (i.e., $\Phi = \exp(\mathrm{Lie\ series})$)
2. $\Phi = 1 + \frac{1}{24}[x, y] + \dots$
3. Pentagon equation:

$$\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} = \Phi^{1,2,34}\Phi^{12,3,4}$$

$\Phi^{1,2,3} = \Phi(t_{1,2}, t_{2,3})$, $\Phi^{12,3,4} = \Phi(t_{12,3}, t_{3,4})$, etc.

Pentagon equation



Importance of associators

- in knot theory (finite type invariants)
- in number theory (multiple zeta values)
- in quantization (Tamarkin's approach)
- in Lie theory (Etingof–Kazhdan quantization of Lie bialgebras)

Theorem (Drinfeld, Le–Murakami)

The pentagon equation admits an explicit solution over $\mathbb{C}$:

$$\Phi(x,y) = \sum_{k,m} \left(\frac{i}{2\pi}\right)^{m_1+\cdots+m_k} \zeta(m_1,\ldots,m_k) x^{m_1-1} y x^{m_2-1} y \ldots x^{m_k-1} y$$
$$+ \quad \text{regularized terms}$$

where

$$\zeta(m_1,\ldots,m_k) = \sum_{n_1>\cdots>n_k>0} \frac{1}{n_1^{m_1}\ldots n_k^{m_k}}$$

Note: $\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}$

Theorem (Drinfeld)

The pentagon equation admits solutions over $\mathbb{Q}$

No explicit formulas available.

Definition

The (homogeneous) Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ consists of all $\phi \in \mathrm{free\ Lie}\,(x, y)$, $\deg(\phi) \geq 3$, satisfying

$$\phi^{1,2,3} + \phi^{1,23,4} + \phi^{2,3,4} = \phi^{12,3,4} + \phi^{1,2,34}.$$

Theorem (Drinfeld)

GRT $= exp(\mathfrak{grt})$ *acts freely and transitively on the set of Drinfeld associators.*

Associators $\Longrightarrow$ KV

Let

$$\psi(\Phi x \Phi^{-1}, y) = \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \Phi(\tau x, \tau y) \right)_{\tau=1} \Phi(x,y)^{-1}.$$

ψ is an element of the (inhomogeneous) Grothendieck–Teichmüller Lie algebra $\mathfrak{gt}$.

Recall: $\mathrm{ch}(x,y) = \ln(e^x e^y)$.

Theorem (Enriquez, Torossian, Podkopaeva, Severa, A.A.)

$A(x,y) = \psi(-\mathrm{ch}(x,y), x)$

$B(x,y) = \psi(-\mathrm{ch}(x,y), y) - \frac{1}{2}\mathrm{ch}(x,y)$

solves KV.

Uniqueness problem

Definition

The Kashiwara–Vergne Lie algebra krv consists of all pairs $a, b \in$ free Lie (x, y), such that

- $[x, a] + [y, b] = 0$
- $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_x \circ \partial_x a + \mathrm{ad}_y \circ \partial_y b) = 0$ for all $\mathfrak{g}$

Theorem (Torossian, A.A.)

$$\phi \mapsto (\phi(-x - y, x), \phi(-x - y, y))$$

is an injection of Lie algebras $\mathfrak{grt} \rightarrowtail \mathrm{krv}$

Conjectures

Conjecture: $\mathfrak{grt} \cong \mathrm{krv}$

numerical evidence up to degree 16.

deg	1	2	3	4	5	6	7	8	9	10	11	...
dim	0	0	0	0	0	0	0	1	1	0	1	...

Conjectures

Number theory properties of associators $\Longrightarrow$ Lie algebra dmr_0 (Racinet)

Theorem (Furusho)

$$\mathfrak{grt} \rightarrowtail \mathrm{dmr}_0$$

Conjecture: $\mathfrak{grt} \cong \mathrm{dmr}_0$

numerical evidence up to degree 19

Theorem (Schneps)

$$\mathrm{dmr}_0 \rightarrowtail \mathrm{krv}$$

Conjecture: $\mathrm{dmr}_0 \cong \mathrm{krv}$

THANK YOU!