On positive reachability of time-variant systems

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1 Preliminaries

We introduce here the main concepts, recall definitions and facts, and set notation. For more information on positive continuous-time and discrete-time systems, the reader is referred to e.g. [FR], and for information on time scales calculus, to e.g. [BP].

1.1 Positive math

By $\mathbb{R}$ we shall denote the set of all real numbers, by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of natural numbers (without 0). We shall also need the set of nonnegative real numbers, denoted by $\mathbb{R}_+$ and the set of nonnegative integers $\mathbb{Z}_+$, i.e. $\mathbb{N} \cup \{0\}$. Similarly, $\mathbb{R}_+^k$ will mean the set of all column vectors in $\mathbb{R}^k$ with nonnegative components and $\mathbb{R}_+^{k\times p}$ will consist of $k \times p$ real matrices with nonnegative elements. If $A \in \mathbb{R}_+^{k\times p}$ we write $A \geq 0$ and say that A is *nonnegative*. A nonnegative matrix A will be called *positive* if at least one of its elements is greater than 0. Then we shall write $A > 0$.

A positive column or row vector is called *monomial* if one of its components is positive and all the other are zero. A monomial column in $\mathbb{R}_+^n$ has the form αe_k for some $\alpha > 0$ and $1 \leq k \leq n$, where e_k denotes the column with 1 at the kth position and other elements equal 0. Then we say that the column is *k-monomial*. An $n \times n$ matrix A is called *monomial* if all columns and rows of A are monomial. Then A is invertible and its inverse is also positive. Moreover, we have the following important fact.

Proposition 1.1. *A positive matrix A has a positive inverse if and only if A is monomial.*

It will be convenient to extend the set of all real numbers adding one element. It will be denoted by ∞ and will mean the positive infinity. We set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $\bar{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. If $a \in \mathbb{R}$ then we define $a + \infty = \infty$. Moreover, for $a \in \mathbb{R}$ and $a > 0$ we set $a/0 = \infty$ and $a/\infty = 0$. Of course $\infty > 0$. If a matrix A has elements from $\bar{\mathbb{R}}$, then the notions of nonnegativity and positivity have the same meanings as before and are denoted in the same way. Addition of such matrices is defined in the standard way, but we shall not need multiply or invert such matrices.

A subset C of $\mathbb{R}^n$ is called a *(positive) cone* if for any $\alpha \in \mathbb{R}_+$ and any $x \in C$, $\alpha x \in C$. It is clear that $\mathbb{R}^n_+$ is a cone.

1.2 Calculus on time scales

Calculus on time scales is a generalization of the standard differential calculus and the calculus of finite differences.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. In particular $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and $\mathbb{T} = q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ for $q > 1$ are time scales. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. If $t_0, t_1 \in \mathbb{T}$, then $[t_0, t_1]_{\mathbb{T}}$ denotes the intersection of the ordinary closed interval with $\mathbb{T}$. Similar notation is used for open, half-open or infinite intervals.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ when $\sup \mathbb{T}$ is finite; the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ when $\inf \mathbb{T}$ is finite; the *forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$; the *backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$ by $\nu(t) := t - \rho(t)$.

If $\sigma(t) > t$, then t is called *right-scattered*, while if $\rho(t) < t$, it is called *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

The time scale $\mathbb{T}$ is *homogeneous*, if μ and ν are constant functions. When $\mu \equiv 0$ and $\nu \equiv 0$, then $\mathbb{T} = \mathbb{R}$ or $\mathbb{T}$ is a closed interval (in particular a half-line). When μ is constant and greater than 0, then $\mathbb{T} = \mu\mathbb{Z}$.

Let $\mathbb{T}^k := \{t \in \mathbb{T} : t \text{ is nonmaximal or left-dense}\}$. Thus $\mathbb{T}^k$ is got from $\mathbb{T}$ by removing its maximal point if this point exists and is left-scattered.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The *delta derivative of f at t*, denoted by $f^{\Delta}(t)$, is the real number with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ such that

$$|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. If $f^{\Delta}(t)$ exists, then we say that *f is delta differentiable at t*. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

Example 1.2. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$. If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h}$. If $\mathbb{T} = q^{\mathbb{N}}$, then $f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. If f is continuous, then it is rd-continuous.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Let $a, b \in \mathbb{T}$. Then the *delta integral* of f on the interval

$[a,b)_{\mathbb{T}}$ is defined by

$$\int_a^b f(\tau)\Delta\tau := \int_{[a,b)_{\mathbb{T}}} f(\tau)\Delta\tau := F(b) - F(a).$$

It is more convenient to consider the half-open interval $[a,b)_{\mathbb{T}}$ than the closed interval $[a,b]_{\mathbb{T}}$ in the definition of the integral. If b is a left-dense point, then the value of f at b would not affect the integral. On the other hand, if b is left-scattered, the value of f at b is not essential for the integral (see Example 1.3). This is caused by the fact that we use delta integral, corresponding to the forward jump function.

Riemann and Lebesgue delta integrals on time scales have been also defined (see e.g. [G]). It can be shown that every rd-continuous function has an antiderivative and its Riemann and Lebesgue integrals agree with the delta integral defined above.

We have a natural property:

$$\int_a^b f(\tau)\Delta\tau = \int_a^c f(\tau)\Delta\tau + \int_c^b f(\tau)\Delta\tau$$

for any $c \in (a,b)_T$. Moreover, if f is rd-continuous, $f(t) \geq 0$ for all $a \leq t < b$ and $\int\limits_a^b f(\tau)\Delta\tau = 0$, then $f \equiv 0$.

Example 1.3. If $\mathbb{T} = \mathbb{R}$, then $\int\limits_a^b f(\tau)\Delta\tau = \int\limits_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int\limits_a^b f(\tau)\Delta\tau = \sum\limits_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

1.3 Linear systems on time scale

Let us consider the system of delta differential equations on a time scale $\mathbb{T}$:

$$x^{\Delta}(t) = A(t)x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $A(t)$ is a $n \times n$ matrix. We assume that A is continuous on $\mathbb{T}$.

Proposition 1.4. *Equation* (1) *with initial condition* $x(t_0) = x_0$ *has a unique forward solution defined for all* $t \in [t_0, +\infty)_{\mathbb{T}}$.

The *matrix exponential function* (at t_0) for A is defined as the unique forward solution of the matrix differential equation $X^{\Delta}(t) = A(t)X(t)$, with the initial condition $X(t_0) = I$. Its value at t is denoted by $e_A(t, t_0)$.

Proposition 1.5. *The following properties hold for every $t, s, r \in \mathbb{T}$ such that $r \leq s \leq t$:*
i) $e_A(t,t) = I$;
ii) $e_A(t,s)e_A(s,r) = e_A(t,r)$;

Let us consider now a nonhomogeneous system

$$x^{\Delta}(t) = A(t)x(t) + f(t) \tag{2}$$

where A is continuous and f are rd-continuous.

Theorem 1.6. *Let $t_0 \in \mathbb{T}$. System (2) for the initial condition $x(t_0) = x_0$ has a unique forward solution of the form*

$$x(t) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A(t,\sigma(\tau))f(\tau)\Delta\tau. \tag{3}$$

2 Positive control systems

Let $n \in \mathbb{N}$ be fixed. From now on we shall assume that the time scale $\mathbb{T}$ consists of at least $n+1$ elements.

Let us consider a linear control system, denoted by Σ, and defined on the time scale $\mathbb{T}$:

$$x^{\Delta}(t) = A(t)x(t) + B(t)u(t) \tag{4}$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, A and B are continuous.

We assume that the control u is a piecewise continuous function defined on some interval $[t_0, t_1)_{\mathbb{T}}$, depending on u, where $t_0 \in \mathbb{T}$ and $t_1 \in \mathbb{T}$ or $t_1 = \infty$. We shall assume that at each point $t \in [t_0, t_1)_{\mathbb{T}}$, at which u is not continuous, u is right-continuous and has a finite left-sided limit if t is left-dense. This allows to solve (4) step by step. Moreover, for a finite t_1 we can always evaluate $x(t_1)$. For t_1 being left-scattered we do not need the value of u at t_1, and for a left-dense t_1 we just take a limit of $x(t)$ at t_1.

Definition 2.1. We say that system Σ is *positive* if for any $t_0 \in \mathbb{T}$, any initial condition $x_0 \in \mathbb{R}^n_+$, any control $u : [t_0, t_1)_{\mathbb{T}} \to \mathbb{R}^m_+$ and any $t \in [t_0, t_1]_{\mathbb{T}}$, the solution x of (4) satisfies $x(t) \in \mathbb{R}^n_+$.

By the separation principle we have the following characterization.

Proposition 2.2. *The system Σ is positive if and only if $e_A(t,t_0) \in \mathbb{R}^{n\times n}_+$ for every $t, t_0 \in \mathbb{T}$ such that $t \geq t_0$, and $B(t) \in \mathbb{R}^{n\times m}_+$ for $t \in \mathbb{T}$.*

3 Reachability

If Σ is a positive system, then for a nonnegative initial condition x_0 and a nonnegative control u, the trajectory x stays in $\mathbb{R}^n_+$. One may be interested in properties of the reachable sets of the system. For simplicity we assume that the initial condition is $x_0 = 0$. Let $x(t_1, t_0, 0, u)$ mean the trajectory of the system corresponding to the initial condition $x(t_0) = 0$ and the control u, and evaluated at time t_1. We shall define various controllability properties.

Definition 3.1. Let $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. The *positive reachable set* (from 0) of the system Σ on the interval $[t_0, t_1]_{\mathbb{T}}$ is the set $\mathcal{R}_+^{[t_0,t_1]}$ consisting of all $x(t_1, t_0, 0, u)$, where u is a nonnegative control on $[t_0, t_1)_{\mathbb{T}}$.

The system Σ is *positively reachable on* $[t_0, t_1]_{\mathbb{T}}$ if $\mathcal{R}_+^{[t_0,t_1]} = \mathbb{R}_+^n$.

To study positive reachability let us introduce a modified Gram matrix related to the control system.

Definition 3.2. Let $M \subseteq \{1, \ldots, m\}$ and $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. For each $k \in M$ let S_k be a subset of $[t_0, t_1)_{\mathbb{T}}$ that is a union of finitely many disjoint intervals of $\mathbb{T}$ of the form $[\tau_0, \tau_1)_{\mathbb{T}}$, and let $\mathcal{S}_M = \{S_k : k \in M\}$. By *the Gram matrix of system* (4) *corresponding to* t_0, t_1, M *and* $\mathcal{S}_M$ we mean the matrix

$$W := W_{t_0}^{t_1}(M, \mathcal{S}_M) := \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau)) b_k(\tau) b_k^T(\tau) e_A(t_1, \sigma(\tau))^T \Delta\tau. \tag{5}$$

Then we have the following characterization:

Theorem 3.3. *Let $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. System* (4) *is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ iff there are $M \subseteq \{1, \ldots, m\}$ and the family $\mathcal{S}_M = \{S_k : k \in M\}$ of subsets of $[t_0, t_1]_{\mathbb{T}}$ such that the matrix $W = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ is monomial.*

Proof. "⇐" Let $\bar{x} \in \mathbb{R}_+^n$. By $\tilde{e}_1, \ldots, \tilde{e}_m$ we denote the vectors of the standard basis in $\mathbb{R}^m$. Define control $u : [t_0, t_1) \to \mathbb{R}_+^m$ by $u(\tau) = \sum_{k \in M} u_k(\tau) \tilde{e}_k$, where $u_k(\tau) = b_k(\tau)^T e_A(t_1, \sigma(\tau))^T W^{-1} \bar{x}$ for $t \in S_k$ and $u_k(\tau) = 0$ for $t \notin S_k$. The control u is nonnegative and

$$\begin{aligned} x(t_1) &= \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) B(\tau) u(\tau) \Delta\tau = \sum_{k \in M} \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) b_k(\tau) u_k(\tau) \Delta\tau \\ &= \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau)) b_k(\tau) b_k^T(\tau) e_A(t_1, \sigma(\tau))^T W^{-1} \bar{x} \Delta\tau = \bar{x}. \end{aligned}$$

Thus (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$.
"⇒" Positive reachability implies that all the vectors $e_1, \ldots, e_n$ can be reached using nonnegative controls. Let us fix some e_i. Then there is a piecewise continuous nonnegative control $u = (u_1, \ldots, u_m)$ on $[t_0, t_1)_{\mathbb{T}}$ such that

$$e_i = \sum_{j=1}^{m} \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) b_j(\tau) u_j(\tau) \Delta\tau.$$

Since all the integrals in the sum are nonnegative, for some k_i the integral

$$\int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) b_{k_i}(\tau) u_{k_i}(\tau) \Delta\tau$$

is an i-monomial vector. Then for every $\tau \in [t_0, t_1)_{\mathbb{T}}$ the vector $e_A(t_1, \sigma(\tau))b_{k_i}(\tau)u_{k_i}(\tau)$ is either i-monomial or 0. Let T_i be the set of all τ for which $e_A(t_1, \sigma(\tau))b_{k_i}(\tau)u_{k_i}(\tau)$ is i-monomial. Then for $\tau \in T_i$ the matrix

$$e_A(t_1, \sigma(\tau))b_{k_i}(\tau)b_{k_i}^T(\tau)e_A(t_1, \sigma(\tau))^T$$

is diagonal with the only nonzero element at the ith place. The same is true for the matrix $\int_{T_i} e_A(t_1, \sigma(\tau))b_{k_i}(\tau)b_{k_i}^T(\tau)e_A(t_1, \sigma(\tau))^T \Delta\tau$. This implies that the matrix

$$C := \sum_{i=1}^{n} \int_{T_i} e_A(t_1, \sigma(\tau))b_{k_i}(\tau)b_{k_i}^T(\tau)e_A(t_1, \sigma(\tau))^T \Delta\tau$$

is monomial (and diagonal). Let M consist of all k_i for $i = 1, \dots, n$. Observe that if $k_i = k_j$ for $i \neq j$, then $T_i \cap T_j = \emptyset$. Define $S_k = \bigcup_{k_i=k} T_i$ and let $\mathcal{S}_M = \{S_k : k \in M\}$. Then

$$C = \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau))b_k(\tau)b_k^T(\tau)e_A(t_1, \sigma(\tau))^T \Delta\tau = W_{t_0}^{t_1}(M, \mathcal{S}_M),$$

so $W_{t_0}^{t_1}(M, \mathcal{S}_M)$ is monomial. □

Corollary 3.4. *If the ordinary Gram matrix*

$$W_{t_0}^{t_1} = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau))B(\tau)B^T(\tau)e_A(t_1, \sigma(\tau))^T \Delta\tau$$

is monomial, then system (4) *is positively reachable on* $[t_0, t_1]_{\mathbb{T}}$.

Proof. Observe that $W_{t_0}^{t_1} = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ for $M = \{1, \dots, m\}$ and $S_k = [t_0, t_1)_{\mathbb{T}}$ for all $k \in M$. Thus positive reachability follows from Theorem 3.3. □

Remark 3.5. The condition that $W_{t_0}^{t_1}$ is monomial is not necessary for positive reachability on $[t_0, t_1]_{\mathbb{T}}$. Consider the system

$$x^\Delta = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u \tag{6}$$

on $\mathbb{T} = \mathbb{Z}$. Choose $t_0 = 0$ and $t_1 = 2$. System (6) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$. Indeed, let $M = \{1\}$ and $S_1 = [0, 2)_{\mathbb{T}}$. Then

$$\begin{aligned} W =& b_1 b_1^T + (I + A)b_1 b_1^T (I + A)^T \\ =& \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

is a monomial matrix. However

$$W_{t_0}^{t_1} = BB^T + (I + A)BB^T(I + A)^T = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}$$

is not monomial.

Corollary 3.6. *If there exists $M \subseteq \{1,\dots,m\}$ such that the matrix $W_{t_0}^{t_1}(M) = \int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))\tilde{B}(\tau)\tilde{B}^T(\tau)e_A(t_1,\sigma(\tau))^T\Delta\tau$ is monomial, where $\tilde{B}$ is a submatrix of B consisting of column $b_k, k \in M$, then system (4) is positively reachable on $[t_0,t_1]_{\mathbb{T}}$.*

Proof. Observe that $W_{t_0}^{t_1}(M) = W_{t_0}^{t_1}(M,\mathcal{S}_M)$ where $S_k = [t_0,t_1)_{\mathbb{T}}$ for all $k \in M$. Thus positive reachability follows from Theorem 3.3. □

Remark 3.7. The condition that $W_{t_0}^{t_1}(M)$ is monomial is not necessary for positive reachability on $[t_0,t_1]_{\mathbb{T}}$. Let the time scale $\mathbb{T} = \{0\} \cup [1,2] \cup \{3\}$. Consider the system

$$x^{\Delta} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u. \tag{7}$$

The system is positively reachable on $[0,3]_{\mathbb{T}}$. Indeed, let $M = \{1\}$ and let $S_1 = [0,1)_{\mathbb{T}} \cup [2,3)_{\mathbb{T}}$. Then

$$\begin{aligned} W = &\int_{[0,1)_{\mathbb{T}}} e_A(3,\sigma(\tau))BB^Te_A(3,\sigma(\tau))^T\Delta\tau \\ &\qquad + \int_{[2,3)_{\mathbb{T}}} e_A(3,\sigma(\tau))BB^Te_A(3,\sigma(\tau))^T\Delta\tau \\ = &\begin{pmatrix} 0 & 0 \\ 0 & e^{-2} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2} \end{pmatrix} \end{aligned}$$

is monomial. Observe that we remove here the points t with $\mu(t) = 0$. This is essential in order to get a monomial matrix. To calculate the full Gram matrix we have to add to W the following matrix

$$\int_{[1,2)} e_A(3,\sigma(\tau))BB^Te_A(3,\sigma(\tau))^T d\tau.$$

Its off-diagonal elements are equal to $\int_1^2 (3-\tau)e^{-2(3-\tau)}d\tau$. Since they are positive, $W_{t_0}^{t_1}(M)$ is not monomial.

From the general characterization of positive reachability presented in Theorem 3.3 we can deduce more concrete results for particular time scales. For $\mathbb{T} = \mathbb{R}$ we get very restrictive conditions for positive reachability. The following result was first obtained in [CA] for constant matrices A and B.

Proposition 3.8. *Let $\mathbb{T} = \mathbb{R}$ and $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$. Let A and B be analytic. System (4) is positively reachable on $[t_0,t_1]$ iff A is diagonal and B contains an $n \times n$ submatrix that is monomial for almost every $t \in [t_0,t_1]$ (so $m \geq n$).*

Proof. "$\Leftarrow$" Let $\tilde{B}(t)$ denote the monomial submatrix of $B(t)$ and let the indices of columns of $\tilde{B}(t)$ form the set M. Then $\tilde{B}(t)\tilde{B}^T(t)$ is a diagonal matrix with all the diagonal elements being positive and so is

$$W_{t_0}^{t_1}(M) = \int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))\tilde{B}(t)\tilde{B}^T(t)e_A(t_1,\sigma(\tau))^T\Delta\tau.$$

Thus $W_{t_0}^{t_1}(M)$ is monomial, so system (4) is positively reachable by Corollary 3.6. Observe that the proof of this implication works for all time scales.

"$\Rightarrow$" Assume that the system is positively reachable on $[t_0, t_1]$. From Theorem 3.3 it follows that for some set M and some family $\mathcal{S}_M$ the Gram matrix $W = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ is monomial. Let jth column of W be i-monomial. Then for some $k \in M$ and for τ from some subinterval of $[t_0, t_1)$ the jth column of the matrix $e_A(t_1,\tau)b_k(\tau)b_k^T(\tau)(e_A(t_1,\tau))^T$ is i-monomial. Let $c(\tau) = e_A(t_1,\tau)b_k(\tau)$. Since the jth column of the matrix $c(\tau)c(\tau)^T$ is i-monomial, then $c(\tau)$ must be j-monomial and eventually $i = j$. This means that at least one column of $e_A(t_1,\tau)$ must be i-monomial. As the exponential matrix is invertible such a column must be unique. This implies that $b_k(\tau)$ is monomial. Moreover the i-monomial column of $e_A(t_1,\tau)$ must be its ith column. Otherwise we would get 0 on the diagonal of the analytic exponential matrix for all τ from some interval, which is impossible. Thus $e_A(t_1,\tau)$ is diagonal on some interval, which means that $A(t)$ is also diagonal. Now to get all n monomial columns in W we need n different monomial columns $b_k(t)$. Thus $B(t)$ contains an $n \times n$ monomial submatrix. □

For discrete homogeneous time scales the conditions for positive reachability are much less restrictive.

Proposition 3.9. *Let $\mathbb{T} = \mu\mathbb{Z}$ for a constant $\mu > 0$. Let A and B be constant. Let $t_0 \in \mathbb{T}$ and $t_1 = t_0 + k\mu$ for some $k \in \mathbb{N}$. System (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ iff the matrix $[B, (I+\mu A)B, \ldots, (I+\mu A)^{k-1}B]$ contains a monomial submatrix.*

Proof. "$\Leftarrow$" Observe that $x(t_1) = \sum_{i=0}^{k-1}\sum_{j=1}^{m}(I+\mu A)^i b_j u_j(k-1-i)$. If $(I+\mu A)^i b_j = \gamma e_s$ for some $\gamma > 0$, then setting $u_j(k-1-i) = 1/\gamma$ and all other components and values at different times putting to 0 we get $x(t_1) = e_s$. This means positive reachability on $[t_0, t_1]_{\mathbb{T}}$.
"$\Rightarrow$" By Theorem 3.3 positive reachability implies existence of a set M and subsets S_k of $[t_0, t_1]$ for $k \in M$ such that the matrix

$$W = \sum_{k\in M}\int_{S_k} e_A(t_1,\sigma(\tau))b_k b_k^T e_A(t_1,\sigma(\tau))^T \Delta\tau$$

is monomial. Moreover

$$\int_{S_k} e_A(t_1,\sigma(\tau))b_k b_k^T e_A(t_1,\sigma(\tau))^T \Delta\tau = \sum_{t\in S_k}(I+\mu A)^{(t_1-t-\mu)/\mu} b_k b_k^T ((I+\mu A)^{(t_1-t-\mu)/\mu})^T \mu.$$

This implies that for every $i = 1, \ldots, n$ there are $k \in M$, $t \in S_k$ and $0 \le j \le n$ such that the jth column of $(I+\mu A)^{(t_1-t-\mu)/\mu} b_k b_k^T((I+\mu A)^{(t_1-t-\mu)/\mu})^T$ is i-monomial. This means that the column $(I+\mu A)^{(t_1-t-\mu)/\mu} b_k$ is i-monomial. But this column is one of the columns of the matrix $[B, (I+\mu A)B, \ldots, (I+\mu A)^{k-1}B]$. □

Proposition 3.9 may be extended to nonhomogeneous discrete time scales and nonconstant matrices A and B.

Proposition 3.10. *Assume that $\mu(t) > 0$ for all $t \in \mathbb{T}$, $t_0 \in \mathbb{T}$ and $t_1 = \sigma^k(t_0)$. System* (4) *is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ iff the matrix*

$$[B(\sigma^{k-1}(t_0)), (I + \mu(\sigma(t_0))A(\sigma(t_0)))B(\sigma^{k-2}(t_0)),\\ (I + \mu(\sigma^2(t_0))A(\sigma^2(t_0)))(I + \mu(\sigma(t_0))A(\sigma(t_0)))B(\sigma^{k-3}(t_0)), \dots,\\ (I + \mu(\sigma^{k-1}(t_0))A(\sigma^{k-1}(t_0))) \dots (I + \mu(\sigma(t_0))A(\sigma(t_0)))B(t_0)]$$

contains a monomial submatrix.

The proof is similar to the proof of Proposition 3.9, but we have to take into account that the exponential matrix is no longer a power of $I + \mu A$ for a constant μ but rather a product of such terms with possibly different values of μ and A. This criterion may be used for systems on $\mathbb{T} = q^{\mathbb{N}}$.

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