On the integral locus of symmetric algebras

Belén López, Agustín Marcelo, Félix Marcelo and César Rodríguez
Departamento de Matemáticas, Campus de Tafira,
Universidad de Las Palmas de Gran Canaria, Spain
e-mail: {blopez, amarcelo, cesar}@dma.ulpgc.es

Abstract

The integral locus of symmetric algebras is shown to be an open set of the Zariski topology. Then this result is used to give a primality criterion for ideals of polynomial rings generated by a family of linear forms and to study the Rees algebra of a submodule of a free module.

1 Introduction

Let R be a Noetherian domain, let M be a finite torsion-free R-module and let $S(M)$ be the symmetric algebra of M.

In this note we prove that the set of points $\mathfrak{p} \in SpecR$ such that $S(M_{\mathfrak{p}})$ is an integral domain (the integral locus of $S(M)$) is an open set of the Zariski topology in $SpecR$. We use this result to give a primality criterion for ideals of polynomial rings over R generated by a family of linear forms. As a consequence of this result we can show that for every submodule M of a free R-module there exists an open set U of $SpecR$ such that for all $\mathfrak{p} \in U$ then $\mathcal{R}(M_{\mathfrak{p}})$ is a symmetric algebra.

2 Integral locus

Let R be a Noetherian domain, let N be a finite R-module and let $S(N)$ be the symmetric algebra of N.

Theorem 1 *The set of points $\mathfrak{p} \in SpecR$ such that $S(N_{\mathfrak{p}})$ is an integral domain is an open set of the Zariski topology of SpecR.*

Proof. Let T be the R-torsion of $S(N)$. By virtue of [3] T is a prime ideal. Hence T is a finitely-generated $S(N)$-module and $Ass_{S(N)}T$ is a finite set. Moreover by [2, (9.A) Proposition] we have

$$Ass_R T =^a \phi(Ass_{S(N)} T')$$

where ϕ is the rings homomorphism $\phi : R \longrightarrow S(N)$. Now we apply [1, Chapitre IV, §1, 3, Corr. 1] to get that

$$Ass_R T \subset Supp_R T$$

and both sets have the same minimal elements.

Since $Ass_R T$ is a finite set we conclude that $Supp_R T$ is a closed set in $SpecR$. Then $SpecR \setminus Supp_R T$ satisfies the statement. ■

3 Applications

Let $f_1, ..., f_r$ be a family of linear forms of $S = R[x_1, ..., x_d]$ and set $J = (f_1, ..., f_r) \cdot S$.

Corollary 2 *There exists an open set V of the Zariski topology of $SpecR$ such that $J_{\mathfrak{p}}$ is a prime ideal of S if and only if $\mathfrak{p} \in V$.*

Proof. Set $A = Rf_1 + ... + Rf_r$ and $F = Rx_1 \oplus ... \oplus Rx_d$. From the exact sequence

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$$

we get

$$0 \longrightarrow J \longrightarrow S \longrightarrow S(B) \longrightarrow 0,$$

where $S(B)$ is the symmetric algebra of B. Let $\mathfrak{p}$ be a prime ideal of R. Then $J_{\mathfrak{p}}$ is a prime ideal of $S(N_{\mathfrak{p}})$ if and only if $S(B_{\mathfrak{p}})$ is an integral domain and we get the desired result from Theorem 1. ■

Corollary 3 *Let M be a submodule of a free R-module of finite rank and let $\mathcal{R}(M)$ be its Rees algebra. Then there exists an open set U of $SpecR$ such that for all $\mathfrak{p} \in U$ then $\mathcal{R}(M_{\mathfrak{p}}) = S(M_{\mathfrak{p}})$.*

Proof. As it is well-known, $\mathcal{R}(M)$ is a factor ring of a polynomial ring modulo a prime ideal, i. e., we have an exact sequence

$$0 \longrightarrow Q \longrightarrow R[x_1, ..., x_d] \longrightarrow \mathcal{R}(M) \longrightarrow 0.$$

By using the previous result for all $\mathfrak{p} \in V$ it follows that $Q_{\mathfrak{p}}$ is generated by linear elements. Therefore $\mathcal{R}(M_{\mathfrak{p}}) = S(M_{\mathfrak{p}})$. ■

Let R be a factorial domain and let N be a torsion-free finite R-module. According to [3] we denote $B(N)$ the bidual of $S(N)$ over R, i.e., $B(N) = \bigoplus_{t \geq 0} S_t(N)^{**}$.

Proposition 4 *Assume that $B(N)$ is a finitely-generated R-algebra. Then the set of points $\mathfrak{p} \in SpecR$ such that $S(N_{\mathfrak{p}})$ is factorial is an open set of the Zariski topology of $SpecR$.*

Proof. Set $T' = B(N)/S(N)$. By hypothesis T' is a finitely generated $S(N)$-module. Hence $Ass_{S(N)}T\,'$is a finite set. By virtue of [2, (9.A) Proposition] we have, as in the Theorem 1, that

$$Ass_R T' =^a \phi(Ass_{S(N)}T'').$$

We may now follow the same process as in Theorem 3 to get the desired result. ■

References

[1] Bourbaki, N. *Algèbre Commutative* , Hermann, Paris, 1961.

[2] Matsumura, H.*Commutative Algebra,* Benjamin, New York, 1970.

[3] Vasconcelos, W. *Arithmetic of Blowup Algebras*, Cambridge University Press, Cambridge, U.K., 1994.