An analogue of the Variational Principle for group actions

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Abstract

A classical discrete-time dynamical system consists of a non-empty set X endowed with a structure and a cyclic group or a cyclic semigroup $G =< f >$ generated by a map $f : X \to X$ which preserves the structure of X. Topological dynamical system consists of topological space X and continuous map $f : X \to X$.

It is well known that a continuous map $f : X \to X$ of a compact metric space X determines a f-invariant measure μ and one can define a measure-theoretic entropy $h_\mu(f)$ with respect to μ. A relationship between topological entropy and measure-theoretic entropy of a map $f : X \to X$ is established by the Variational Principle, which asserts that

$$h_{top}(f) = \sup\{h_\mu(f) : \mu \in M(X, f)\},$$

where μ ranges over the set $M(X, f)$ of all f-invariant Borel probability measures on X. We obtain a generalized dynamical system by exchanging the cyclic group $G =< f >$, generated by a single map $f : X \to X$, for a finitely generated group of homeomorphisms of X.

Ghys, Langevin and Walczak [**GLW**] define a notion of topological entropy $h_{top}(G, G_1)$ of a finitely generated group G generated by a finite symmetric set G_1 of homeomorphisms of a compact metric space (X, d). If $s(n, \epsilon)$ denotes the maximal cardinality of any $(n, \epsilon)-$separated subset of X then

$$h_{top}(G, G_1) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log(s(n, \epsilon))}{n}.$$

A subset $A \subset X$ is $(n, \epsilon)-$separated if for any two distinct points $x, y \in A$ there exists a map $g \in G$ such that g is a composition of at most n generators from G_1 and $d(g(x), g(y)) \geq \epsilon$.

In general, there are many examples of finitely generated groups of homeomorphisms that do not admit any non-trivial invariant measure.

Brin and Katok [**BK**] consider a compact metric space (X, d) with a continuous mapping $f : X \to X$ preserving a Borel probability non-atomic measure m. They define a local measure entropy $h_m(f, x)$ of f with respect to m at a point $x \in X$ and shows the interrelations between a measure-theoretic entropy and dimension-like characteristics of smooth dynamical systems.

We generalize the notion of local measure entropy for the case of a group of homeomorphisms of a metric space and we introduce an upper local measure entropy $h_\mu^G(x)$ and a lower local measure entropy $h_{\mu,G}(x)$ of a group G with respect to a Borel probability measure μ defined on X. We apply the theory of C-structures, elaborated by Pesin in [**PePi**], [**Pes2**] and [**Pes**], to construct a dimensional type entropy-like invariant and we prove that it coincides with the topological entropy of a group. This approach allows us to obtain an analogue of the variational principle for group actions which is stated in Theorem 1 and Theorem 2.

Theorem 1(in [**Bis**]). Let (G, G_1) be a finitely generated group of homeomorphisms of a compact oriented Riemannian manifold (M, d). Let E be a Borel subset of M, $s \in (0, \infty)$ and μ_v the natural volume measure on M. If

$$h_{\mu_v}^G(x) \le s \quad for\ all\ x \in E \quad then \quad h_{top}((G, G_1), E) \le s.$$

Theorem 2(in [**Bis**]). Let (G, G_1) be a finitely generated group of homeomorphisms of a compact metric space (X, d). Let E be a Borel subset of X and $s \in (0, \infty)$. Denote by μ a Borel probability measure on X. If

$$h_{\mu,G}(x) \ge s \quad for\ all\ x \in E\ and\ \mu(E) > 0 \quad then \quad h_{top}((G, G_1), E) \ge s.$$

Theorem 1 and Theorem 2 are a generalization of Theorem 1 of Ma and Wen [**MW**].

References

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