Primeness, Primitivity and Radicals in Near-rings of Continuous Functions

Geoff Booth
Nelson Mandela Metropolitan University
Email: geoff.booth@nmmu.ac.za

1 Preliminaries

Definition 1.1 A *near-ring* is a triple $(N, +, \cdot)$, where

1. $(N, +)$ is a group;
2. $(N, \cdot)$ is a semigroup;
3. $(x + y)z = xz + yz$ for all $x, y, z \in N$.

In the sequel we will speak of "the near-ring N", rather than "the near-ring $(N, +, \cdot)$" and of xy, rather than $x \cdot y$. If $x0 = 0$ for all $x \in N$, then N is said to be *zero-symmetric*. All near-rings discussed here will be zero-symmetric.

Homomorphisms, *epimorphisms* and *isomorphisms* are defined as they are for rings. *Ideals* of a near-ring N are the kernels of homomorphic mappings of N. The notation $A \lhd N$ means "A is an ideal of N". A *subnear-ring* of N is a subset S of N which is a near-ring with respect to the operations on N.

Definition 1.2 Let $(G, +)$ be a (not necessarily abelian) group, and let $M_0(G) := \{a : G \to G \mid a(0) = 0\}$. Then $M_0(G)$ is a zero-symmetric near-ring with the operations pointwise addition and composition of functions.

We remark that for any group G, $M_0(G)$ is *simple*, i.e. it has only the two trivial ideals.

Definition 1.3 Let $(G, +)$ be a topological group, and let $N_0(G) := \{a : G \to G \mid a \text{ is continuous and } a(0) = 0\}$. Then $N_0(G)$ is a zero-symmetric near-ring with the operations pointwise addition and composition of functions.

In the sequel, G will be a T_0 (and hence completely regular) topological group.

Remarks 1.4

1. Clearly, $N_0(G)$ is a subnear-ring of $M_0(G)$. If the topology on G is discrete, then $N_0(G) = M_0(G)$.

2. While $M_0(G)$ is simple for any group G, $N_0(G)$ seldom is, if G is not discrete. For example, let $P :=$

   $\{a \in N_0(G) \mid a(U) = 0$ for some open set U such that $0 \in U\}$. Then P is frequently a non-trivial ideal of $N_0(G)$.

For all undefined concepts concerning near-rings, we refer to Pilz [9]. For further information on near-rings of continuous functions, we refer to the surveys by Magill [7], [8].

2 Primeness

Definition 2.1 A near-ring N is

1. *0-semiprime* if $A \lhd N$, $A^2 = 0$ implies $A = 0$;

2. *0-prime* if $A, B \lhd N$, $AB = 0$ implies $A = 0$ or $B = 0$;

3. *3-semiprime* if $a \in N$, $aNa = 0$ implies $a = 0$;

4. *3-prime* if $a, b \in N$, $aNb = 0$ implies $a = 0$ or $b = 0$;

5. *equiprime* (*e-prime*) if $a, x, y \in N$, $anx = any$ for all $n \in N$ implies $a = 0$ or $x = y$.

The relationships between the above concepts are equiprime$\Longrightarrow$3-prime$\Longrightarrow$ 0-prime$\Longrightarrow$0-semiprime and 3-prime$\Longrightarrow$3-semiprime$\Longrightarrow$0-semiprime. Equiprimeness is of particular interest from the radical-theoretic viewpoint in that it leads to the only known Kurosh-Amitsur prime radical in the varieties of both zero-symmetric and general near-rings [4].

Proposition 2.2 [5] Let G be a topological group which is 0-dimensional or arcwise connected. Then $N_0(G)$ is equiprime.

Since the discrete topology is 0-dimensional, this implies that $M_0(G)$ is equiprime. This need not be the case for $N_0(G)$ in general, as the following example shows.

Example 2.3 Let $G = \mathbb{R} \times \mathbb{Z}_2$ have the product topology with respect to the usual and discrete topologies on $\mathbb{R}$ and $\mathbb{Z}_2$, respectively. Let $I :=$

$\{a \in N_0(G) \mid a(G) \subseteq \mathbb{R} \times 0\}$ and $J := \{a \in N_0(G) \mid a(\mathbb{R} \times 0) = 0\}$. Then $I, J \lhd N_0(G)$ and $I \cap J \neq 0$. Moreover, $(I \cap J)^2 = 0$, so $N_0(G)$ is not 0-semiprime.

3 Primitivity

A number of notions of primitivity exist in the literature of near-rings, which can be defined in terms of left N-modules, which are defined in a natural way (cf [9]). We recall that the *annihilator* of an N-module G is the set $(0 : G) = \{x \in N \mid xG = 0\}$. G is said to be *faithful* if $(0 : G) = 0$.

Definition 3.1 A left N-module of a near-ring N is said to be:

1. of *type 2* if $NG \neq 0$ and $Ng = 0$ or $Ng = G$, for all $g \in G$.
2. of *type 3* if N is of type 2 and $g_1, g_2 \in G, ng_1 = ng_2$ for all $n \in N$ implies $g_1 = g_2$ (cf Holcombe [6]).

Definition 3.2 A near-ring N is *ν-primitive* ($\nu = 2, 3$) if there exists a faithful left N-module of type ν.

We remark that 2-primitive$\Rightarrow$0-prime, and 3-primitive$\Rightarrow$equiprime. In general the notions of 2-primitive and equiprime are not comparable [4]. However, in the case that N has a unity, the notions of 2-primitive and 3-primitive coincide.

Proposition 3.3 [3] Suppose that G is arcwise connected or 0-dimensional. Then $N_0(G)$ is 3-primitive on G.

4 Radicals

Definition 4.1 Let N be a near-ring and let $I \lhd N$. Then

1. I is a *ν-prime ideal of* N ($\upsilon = 0, 3, e$) if the factor near-ring N/I is υ-prime;
2. the *ν-prime radical of* N, $\mathcal{P}_\upsilon(N)$, is the intersection of the ν-prime ideals of N.

Definition 4.2 Let N be a near-ring and let $I \lhd N$. Then

1. I is a *ν-primitive ideal of* N $(\upsilon = 2, 3)$ if the factor near-ring N/I is υ-primitive;
2. the *ν-Jacobson radical of* N, $\mathcal{J}_\upsilon(N)$, is the intersection of the ν-primitive ideals of N.

The radicals $\mathcal{P}_e$, $\mathcal{J}_2$ and $\mathcal{J}_3$ are known to be Kurosh-Amitsur in the variety of zero-symmetric near-rings [4], [6]. The following inclusion relations exist: $\mathcal{P}_0 \subseteq \mathcal{P}_3 \subseteq \mathcal{P}_e \subseteq \mathcal{J}_3$ The radicals $\mathcal{P}_e$ and $\mathcal{J}_2$ are not comparable.

Proposition 4.3 [**1**] Let G be a disconnected topological group, with open components which are arcwise connected and which contain more than one element. Let H be the component of G which contains 0, $I := \{a \in N_0(G) \mid a(G) \subseteq H\}$ and $J := \{a \in N_0(G) \mid a(H) = 0\}$. Then $\mathcal{P}_0(N_0(G)) = \mathcal{J}_3(N_0(G)) = I \cap J$.

We remark that, because of the inclusion relations among the radicals which we have discussed.above, Proposition 4.3 implies that they all coincide in this case.

5 Sandwich Near-rings

Definition 5.1 Let X and G be a topological space and a topological group respectively, and let $\theta : G \longrightarrow X$ be a continuous map. The *sandwich near-ring* $N_0(G, X, \theta)$ is the set $\{a : X \longrightarrow G \mid a$ is continuous and $a\theta(0) = 0\}$. Addition is pointwise and multiplication is defined by $a \cdot b := a\theta b$. In the sequel, $\theta(0)$ will be denoted x_0. If the topologies on X and G are discrete we denote the near-ring by $M_0(G, X, \theta)$.

In this section we will assume that both G and X have more than one element.

Proposition 5.2 [**5**] Suppose that X is a 0-dimensional. Then $N_0(G, X, \theta)$ is equiprime if and only if θ is injective and $\mathrm{cl}(\theta(G)) = G$, where $\mathrm{cl}(\theta(G))$ denotes the closure of $\theta(G)$ in X.

Proposition 5.3 [1], [3] Let $N_0(G, X, \theta)$ be a sandwich near-ring such that $\theta^{-1}\theta(0) = \{0\}$ and either (1) X is a T_0, 0-dimensional topological space or (2) X is completely regular and G is arcwise connected. Then the following are equivalent:

1. $\mathrm{cl}(\theta(G)) = X$.
2. $N_0(G, X, \theta)$ is 3-semiprime.
3. $N_0(G, X, \theta)$ is 3-prime.
4. $N_0(G, X, \theta)$ is 2-primitive (and is 2-primitive on G in this case).

Proposition 5.4 [1], [3] Let X and G be a completely regular topological space and an arcwise connected topological group, respectively, and let $\theta : G \to X$ be a continuous, injective map. Then the following are equivalent:

1. $\mathrm{cl}(\theta(G)) = X$.
2. $N_0(G, X, \theta)$ is 3-semiprime.
3. $N_0(G, X, \theta)$ is equiprime.
4. $N_0(G, X, \theta)$ is 3-primitive.

Proposition 5.5 [1], [3] Let $N_0(G, X, \theta)$ be a sandwich near-ring, where $\theta^{-1}\theta(0) = \{0\}$ and either (1) X is T_0 and 0-dimensional or (2) X is completely regular and G is arcwise connected. Then $\mathcal{P}_0(N_0(G, X, \theta)) = \mathcal{J}_2(N_0(G, X, \theta)) = (0 : G) = \{a \in N_0(G, X, \theta) : a\theta = 0\}$.

We remark that, in view of the inclusion relations that exist among the radicals, $\mathcal{P}_0(N_0(G, X, \theta)) = \mathcal{J}_2(N_0(G, X, \theta))$ in this case.

Proposition 5.5 [1], [3] Let $N_0(G, X, \theta)$ be a sandwich near-ring, where θ is injective and either (1) X is T_0 and 0-dimensional or (2) X is completely regular and G is arcwise connected. Then $\mathcal{P}_0(N_0(G, X, \theta)) = \mathcal{J}_3(N_0(G, X, \theta)) = (0 : G) = \{a \in N_0(G, X, \theta) : a\theta = 0\}$.

The inclusion relations among the radicals imply that all the radicals which have been discussed will coincide in this case.

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