Alexandrov's theorem in curved manifolds

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The isoperimetric problem

Queen Dido's problem: Minimize the length among all embedded curves in the plane with a given enclosed area.

Generalize this to higher dimensions: Minimize the area of an embedded hypersurface in $\mathbb{R}^n$ among all surfaces that enclose a given volume V.

Theorem. *Round spheres are optimal for the isoperimetric problem. More precisely, if* Ω *is a domain in* $\mathbb{R}^n$ *satisfying* $\text{vol}(\Omega) = \text{vol}(B)$*, then* $|\partial\Omega| \geq |\partial B|$*.*

Symmetrization techniques show that if a minimizer exists it must be a sphere. Existence of a minimizer is a non-trivial issue. A rigorous proof (based on Brunn-Minkowski inequality) was found in the 20th century.

The first variation of surface area

Study the isoperimetric problem using the Calculus of Variations developed by Bernoulli, Euler, and others.

Consider a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Let X be a smooth vector field, and let $\Omega_t = \varphi_t(\Omega)$, where φ_t is the flow generated by X. Then

$$\frac{d}{dt}\text{vol}(\Omega_t)\Big|_{t=0} = \int_{\partial\Omega} \langle X, \nu \rangle$$

and

$$\frac{d}{dt}|\partial\Omega_t|\Big|_{t=0} = \int_{\partial\Omega} H \, \langle X, \nu \rangle.$$

Here, H is the mean curvature of the surface $\partial\Omega$ (can be viewed as an L^2 gradient of the area functional). Note that $H = \lambda_1 + \ldots + \lambda_{n-1}$ where the λ_i's are the principal curvatures of $\partial\Omega$ (i.e. the reciprocals of the curvature radii).

Surfaces with $H = 0$ are minimal surfaces (soap films). The catenoid in $\mathbb{R}^3$ is the simplest example.

Surfaces with constant mean curvature in Euclidean space

Proposition. *Suppose that the first variation of area is zero for variations that leave the enclosed volume unchanged. Then the mean curvature of $\partial\Omega$ is constant: $H = c$.*

The constant c can be viewed as a Lagrange multiplier.

Problem: Analyze hypersurfaces in $\mathbb{R}^n$ with constant mean curvature.

Theorem (Alexandrov 1956). *Let Σ be a hypersurface in $\mathbb{R}^n$ which has constant mean curvature and is embedded (i.e. no self-intersections). Then Σ is a round sphere.*

Wente, Kapouleas: There exist non-trivial examples of constant mean curvature surfaces, but these fail to be embedded.

The proof of Alexandrov's theorem is based on the **method of moving planes**: Reflect Σ across a hyperplane P. Move the hyperplane P until the reflected surface touches the original surface Σ. The reflected surface must coincide with the original surface by the strict maximum principle.

Surfaces with constant mean curvature in Riemannian manifolds

Replace the ambient space $\mathbb{R}^n$ by a curved Riemannian manifold. This question is motivated in part by questions in relativity:

- Christodoulou, Yau (1986): Let $\Sigma \approx S^2$ be a stable surface of constant mean curvature in a 3-manifold (M, g) with positive scalar curvature. Then Σ has nonnegative Hawking mass.

- Huisken, Yau (1996): Let (M, g) be an asymptotically flat 3-manifold which has positive mass (i.e. (M, g) looks Euclidean near infinity). Then (M, g) admits a foliation by constant mean curvature surfaces near infinity. Under mild extra assumptions, this foliation is unique.

- Qing-Tian, Eichmair-Metzger: Optimal uniqueness theorem for large surfaces of constant mean curvature.

- Rigger: Similar results hold when (M, g) is asymptotic to hyperbolic space near infinity. (This corresponds to a universe with negative cosmological constant.)

The Schwarzschild manifold

Recall: The Schwarzschild spacetime describes a static black hole in general relativity.

The **Schwarzschild manifold** is defined as a $\{t = \text{constant}\}$ slice in the Schwarzschild spacetime. The resulting manifold is $M = \{x \in \mathbb{R}^n : |x| \geq 1\}$, and the metric is given by

$$g = (1 + |x|^{2-n})^{\frac{4}{n-2}} \, (dx_1^2 + \ldots + dx_n^2).$$

Note that (M, g) is asymptotic to Euclidean space near infinity. The boundary $\partial M = \{x \in \mathbb{R}^n : |x| = 1\}$ is a minimal surface and is referred to as the **horizon**.

A version of Alexandrov's Theorem for the Schwarzschild manifold

Goal: Classify all surfaces of constant mean curvature in the Schwarzschild manifold (not just the ones near infinity).

Theorem (B. 2011)**.** *Let* Σ *be a closed, embedded, orientable surface in the Schwarzschild manifold with constant mean curvature. Then* Σ *is a sphere of symmetry, i.e.* $\Sigma = \{|x| = r\}$ *for some constant* r*.*

No assumptions on the topology of Σ or the dimension are needed. No stability assumptions are required.

Sketch of proof

Main difficulty: Method of moving planes no longer works!

New approach: Let $M = \{x \in \mathbb{R}^n : |x| \geq 1\}$ and

$$g = (1 + |x|^{2-n})^{\frac{4}{n-2}} \, (dx_1^2 + \ldots + dx_n^2).$$

Consider the static potential

$$f = \frac{1 - |x|^{2-n}}{1 + |x|^{2-n}}$$

and the position vector field

$$X = \sum_{i=1}^n x_i \, \frac{\partial}{\partial x_i}.$$

Note that the $(n+1)$-dimensional Lorentzian metric $-f^2 \, dt^2 + g$ solves the **Einstein vacuum equations**! Moreover, $D_i X_j = f \, g_{ij}$, so X is conformal.

The main theorem is a consequence of the following proposition:

Proposition. *Let Σ be a closed, embedded, orientable hypersurface with positive mean curvature. Then*

$$(n-1)\int_\Sigma f\,d\mu = \int_\Sigma H\,\langle X,\nu\rangle\,d\mu$$

and

$$(n-1)\int_\Sigma \frac{f}{H}\,d\mu \geq \int_\Sigma \langle X,\nu\rangle\,d\mu.$$

Moreover, if equality holds then Σ is a sphere of symmetry.

The first statement follows from the divergence theorem. The second statement is related to a classical inequality of Heintze-Karcher. To prove it, we distinguish two cases:

Case 1: Σ is null-homologous.

Case 2: Σ is homologous to the horizon ∂M.

We will focus here on Case 2, which is the more difficult one.

A geometric evolution equation

We claim:

$$(n-1)\int_\Sigma \frac{f}{H}\,d\mu \geq n\int_\Omega f + A(n).$$

Here, Ω is the region bounded by Σ and the horizon ∂M. $A(n)$ is a positive constant which reflects the contribution of the horizon.

Idea of proof: Deform the surface Σ with speed $-f\,\nu$, where ν is the outward pointing unit normal. This gives a one-parameter family of surfaces Σ_t, $t \geq 0$.

The evolution of the mean curvature is described by the equation

$$\frac{\partial H}{\partial t} = \Delta_\Sigma f + f\sum_i \lambda_i^2 + f\,\mathrm{Ric}(\nu,\nu),$$

where the λ_i's are the principal curvatures.

Key observation: Since the Lorentzian metric $-f^2\,dt^2 + g$ satisfies Einstein's equation, the potential f satisfies

$$(\Delta f)\,g - D^2 f + f\,\mathrm{Ric} = 0.$$

Hence, the evolution of the mean curvature becomes

$$\begin{aligned}\frac{\partial H}{\partial t} &= \Delta_\Sigma f + f\,\sum_i \lambda_i^2 + f\,\mathrm{Ric}(\nu,\nu)\\ &= \Delta f - (D^2 f)(\nu,\nu) + f\,\mathrm{Ric}(\nu,\nu)\\ &- H\,\langle \nabla f, \nu\rangle + f\,\sum_i \lambda_i^2\\ &\geq -H\,\langle \nabla f, \nu\rangle + \frac{1}{n-1}\,f\,H^2.\end{aligned}$$

This allows us to show that the quantity

$$Q(t) = (n-1)\int_{\Sigma_t} \frac{f}{H}\,d\mu - n\int_{\Omega_t} f$$

is **monotone decreasing**!

We next analyze the limit of $Q(t)$ as $t \to \infty$: For t large, Σ_t is close to the horizon, and we obtain

$$\limsup_{t\to\infty} \int_{\Sigma_t} \frac{f}{H}\, d\mu \geq A(n).$$

Consequently,

$$\begin{aligned}&(n-1) \int_{\Sigma} \frac{f}{H}\, d\mu - n \int_{\Omega} f \\ &= Q(0) \geq \limsup_{t\to\infty} Q(t) \geq A(n),\end{aligned}$$

completing the proof of the Theorem.

Concluding remarks:

- The surfaces Σ_t sweep out a **lightlike** hypersurface in the Schwarzschild spacetime (i.e. the surfaces Σ_t move with the speed of light).

- The surfaces Σ_t may not be smooth (due to focal points). To overcome this problem, approximate Σ_t by smooth surfaces.

Constant mean curvature surfaces in warped product manifolds

Goal: Extend the previous analysis to more general spaces with rotational symmetry.

Consider a metric of the form $g = dr^2 + h(r)^2 \, g_{S^{n-1}}$. Assume that the warping function $h : [0, \bar{r}) \to \mathbb{R}$ satisfies the following structure conditions:

(H1) $h'(0) = 0$ and $h''(0) > 0$.

(H2) $h'(r) > 0$.

(H3) The scalar curvature of g is monotone decreasing in r.

Theorem (B. 2011)**.** *Let (M, g) be a rotationally symmetric manifold satisfying conditions (H1)–(H3). Moreover, let Σ be a closed, embedded, orientable hypersurface in (M, g) with constant mean curvature. Then Σ is umbilic.*

This theorem applies, inter alia, to Euclidean space, hyperbolic space, the hemisphere, and the Schwarzschild manifold. The result also applies to the DeSitter-Schwarzschild and Anti-DeSitter-Schwarzschild manifolds (non-zero cosmological constant).

Conversely, Alexandrov's theorem fails if the scalar curvature is not monotone in r:

Theorem (F. Pacard, X. Xu)**.** *Let g be a smooth metric on the ball $B_{\bar{r}}(0) \subset \mathbb{R}^n$ which is rotationally symmetric. Assume that the scalar curvature of g has a strict local extremum somewhere in $B_{\bar{r}}(0)$. Then there exist small spheres with constant mean curvature which are not umbilic.*

Therefore, our result is essentially optimal.

Minimal surfaces in the sphere S^3

We will now discuss a different (though related) problem:

Consider the three-dimensional unit sphere in $\mathbb{R}^4$:

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2+x_2^2+x_3^2+x_4^2 = 1\}.$$

Note: While there are no closed minimal surfaces in $\mathbb{R}^3$, the topology of S^3 allows **closed minimal surfaces**. Examples of minimal surfaces in S^3:

- The equator $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1,\, x_4 = 0\}$.

- The Clifford torus $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}\}$.

Examples of minimal surfaces in S^3:

- H.B. Lawson, Jr. (1970): Given any positive integer g, there exists an embedded minimal surface in S^3 with genus g.

- Karcher, Pinkall, and Sterling (1988): Examples of embedded minimal surfaces in S^3 with genus 3, 5, 6, 7, 11, 19, 73, and 601.

- Further examples constructed recently by Kapouleas-Yang (doubling construction).

Moreover, Lawson has constructed an infinite family of immersed minimal tori in S^3. These surfaces have self-intersections.

The Lawson Conjecture

Motivation: Minimal S^2's in S^3 are unique.

Theorem (Almgren 1966). *Let $\Sigma \approx S^2$ be an immersed minimal surface in S^3. Then Σ is congruent to the equator.*

It turns out that minimal tori in S^3 are unique as well:

Theorem (B. 2012). *The Clifford torus is the only embedded minimal torus in S^3.*

This answers a question posed by H.B. Lawson, Jr., in 1970.

Proof of Lawson's conjecture involves an application of the maximum principle to a function which depends on a **pair of points**.

Proof of Lawson's conjecture (sketch)

Let $F : \Sigma \to S^3$ be an embedded minimal torus and let $\nu : \Sigma \to S^3$ denote the unit normal vector field. We claim that F parametrizes the Clifford torus. Since F is minimal, the sum of the principal curvatures is 0 at each point. If both principal curvatures vanish at x, we say that x is an umbilic point.

Key idea: A minimal torus in S^3 has no umbilic points. Consequently, the ratio

$$Q(x,y) = \sqrt{2}\,\frac{\langle \nu(x), F(y)\rangle}{|A(x)|\,(1 - \langle F(x), F(y)\rangle)}$$

is finite!

Consider a pair of points $(\bar{x}, \bar{y})$ where $Q(x,y)$ is maximal. At the point $(\bar{x}, \bar{y})$, the first derivatives of Q vanish, and the second derivatives form a negative definite matrix.

Proof of Lawson's conjecture (continued)

An intricate calculation shows that $Q(\bar{x}, \bar{y}) \leq 1$. Thus, $Q(x, y) \leq 1$ for all points $x, y \in \Sigma$. An analogous argument gives $Q(x, y) \geq -1$ for all points $x, y \in \Sigma$.

On the other hand, if the gradient of the curvature is non-zero, we can arrange that $|Q(x, y)| > 1$ where x, y are very close to each other. Since $|Q(x, y)| \leq 1$ for all points $x, y \in \Sigma$, the curvature must be constant. This implies that Σ is the Clifford torus.