ON EXPANSIONS OF DIFFERENTIAL OPERATORS IN BANACH SPACES

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It is well-known that the usual theory of partial differential operators expansions (Vishik, Hörmander, Berezansky, Dezin) or that is equivalently, the general theory of boundary value problems has been building in the Hilbert space $L_2(\Omega)$. In this report a starting scheme of theory building for expansions in Banach spaces will be brought and first results of the theory will be obtained.

In a bounded domain $\Omega \subset \mathbb{R}^n$ we consider expansions of operator (initially given in the space $C^\infty(\Omega)$) $\mathcal{L}^+ = \sum\limits_{|\alpha|\le l} a_\alpha(x) D^\alpha$, $D^\alpha = \frac{(-i\partial)^{|\alpha|}}{\partial x^\alpha}$ and its formal adjoint operator $\mathcal{L}^+ \cdot = \sum\limits_{|\alpha|\le l} D^\alpha(a^*_\alpha(x)\ \cdot)$, where $a_\alpha(x) -$ $N \times N^+$-matrix with entries $(a_\alpha)_{ij} \in C^\infty(\bar\Omega)$, $a^*_\alpha(x) -$ adjoint matrix.

For $p > 1$ and $q = p/(p-1)$ we introduce graph norms $\|u\|_{L,p} = \|u\|_{L_p(\Omega)} + \|\mathcal{L}u\|_{L_p(\Omega)}$, $\|u\|_{L,q}$, $\|u\|_{L^+,p}$, $\|u\|_{L^+,q}$. Then we build minimal operators $L_{p0}$, $L_{q0}$, $L^+_{p0}$ and $L^+_{q0}$ with its domains that are understood as the closing $C_0^\infty(\Omega)$ in corresponding graph norms and maximal operators $L_p := (L^+_{q0})^*$, $L_q := (L^+_{p0})^*$, $L^+_p$, $L^+_q$. Each operator $L_{pB} = L_p|_{D(L_{pB})}$ with property $D(L_{p0}) \subset D(L_{pB}) \subset D(L_p)$ is called an expansion (of $L_{p0}$), and the expansion $L_{pB} : D(L_{pB}) \to [L_p(\Omega)]^{N^+} =: B^+_p$ is called **solvable** if there exists its continuous twosided inverse operator $L^{-1}_{pB} : B^+_p \to D(L_{pB})$,
$L_{pB}L^{-1}_{pB} = \mathrm{id}_{B^+_p}$, $L^{-1}_{pB}L_{pB} = \mathrm{id}_{D(L_{pB})}$.

Here as usually one introduces the notion of boundary value problem in the form $L_p u = f$, $\Gamma u \in B$, where subspace $B$ in boundary space $C(L_p) =: D(L_p)/D(L_{p0})$ ($\Gamma : D(L_p) \to C(L_p) -$ factor-mapping) gives a homogenous boundary value problem similar the Hörmander definition. Two Vishik condition of Hipbert case turn to four condition in Banach case: operator $L_{p0}$ has continuous left inverse (condition $(1_p)$) and the same about operators $L_{q0}$ (condition $(1_q)$), $L^+_{p0}$ (condition $(1^+_p)$) and $L^+_{q0}$ (condition $(1^+_q)$). Then we prove the theorems:

**Theorem 1.** The operator $L_{p0}$ has a solvable expansion iff the conditions $(1_p)$ and $(1^+_q)$ are fulfilled.

**Theorem 2.** Under conditions $(1_p)$, $(1^+_p)$ we have decomposition $D(L_p) = D(L_{p0}) \oplus \ker L_p \oplus W_p$, where $W_p -$ some subspace in $D(L_p)$ such that $L_p|_{W_p} : W_p \to \ker L^+_p -$ an isomorphism.

**Theorem 3.** Under conditions $(1_p)$, $(1^+_p)$ any solvable expansion $L_{pB}$ can be decomposed into direct sum $L_{pB} = L_{p0} \oplus L^\partial_{pB}$, where

$L^{\partial}_{pB} : B \to \ker L^{-1}_{p0}-$ some isomorphism.

**Theorem 4.** Under conditions $(1_p)$, $(1^+_p)$ any linear subspace $B \subset C(L_p)$ such that 1) $\Gamma^{-1}_p B \cap \ker L_p = 0$, 2) there exists operator $M_p : \ker L^{-1}_{p0} \to D(L_p)$ with properties: ) $L_p M_p = \mathrm{id}|_{\ker L^{-1}_{p0}}$, ) $\mathrm{Im} M_p \subset \Gamma^{-1}_p B$, generates a well-posed boundary value problem (i.e. a solvable expansion $L_{pB}$ with domain $D(L_{pB}) = \Gamma^{-1} B$).

1. Burskii V.P., *Investigation methods of boundary value problems for general differential equations*, Kiev, Naukova dumka, 2002 (In Russian).

2. Burskii V.P., Miroshnikova A.A., *On expansions of differential operators in Banach spaces.* Nonlinear boundary value problems. Vol.19 (2009), pp. 5-16 (In Russian).