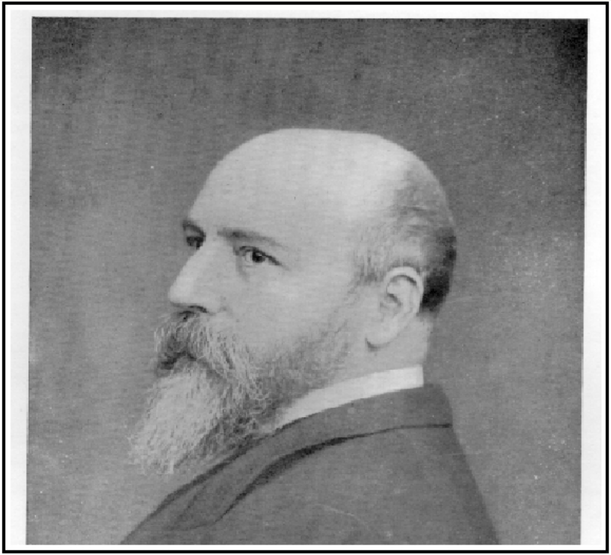
Luigi Cremona (1830 – 1903)



— I —

Basic Features

- $\mathbf{k}$ = a field.
- $\mathbb{A}^n_{\mathbf{k}}$ = affine space of dimension n.
- $\mathrm{Cr}_n(\mathbf{k})$ = **Cremona group** in n variables :
  = group of **birational transformations** of $\mathbb{A}^n_{\mathbf{k}}$

$$\mathrm{Cr}_n(\mathbf{k}) \quad = \quad \mathrm{Bir}(\mathbb{A}^n_{\mathbf{k}})$$

- $\mathbb{P}^n_{\mathbf{k}}$ = projective space of dimension n.

$$\mathrm{Cr}_n(\mathbf{k}) \quad = \quad \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$$

- **Easy fact** .—

$$\mathrm{Cr}_1(\mathbf{k}) = \mathrm{PGL}_2(\mathbf{k}).$$

- For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{k})$

$$\begin{array}{rcl} X \in \mathbb{A}^1_{\mathbf{k}} & \mapsto & \dfrac{aX+b}{cX+d}, \\ [x:y] \in \mathbb{P}^1_{\mathbf{k}} & \mapsto & [ax+by : cx+dy]. \end{array}$$

— From Now on —

$$n = 2$$

- Linear projective transformations $= \mathrm{PGL}_3(\mathbf{k})$
- Monomial transformations $(X, Y) \mapsto (X^a Y^b, X^c Y^d)$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}).$$

There are **indeterminacy points** (like Y/X at the origin).

- $f\colon (X, Y) \mapsto (X + P(Y), Y)$ with $P \in \mathbf{k}(Y)$.

The group $\mathrm{Cr}_2(\mathbf{k})$ is **infinite dimensional**.

Theorem [Lukakikh ; Kollàr and Mangolte].—

The group of birational transformations of $\mathbb{P}^2_{\mathbf{R}}$ *with no real indeterminacy points is dense in* $\mathrm{Diff}^\infty(\mathbb{P}^2(\mathbf{R}))$.

In particular, there are birational transformations of $\mathbb{P}^2_{\mathbf{R}}$ *with rich dynamics.*

— II —

Normal Subgroups and Tits Alternative

Theorem [Herman, Thurston, Mather].—

- *M = smooth, connected, and compact manifold.*
- $\mathrm{Diff}^k(M)$ = *group of diffeomorphisms of class $\mathcal{C}^k$.*
- $\mathrm{Diff}^k_0(M)$ = *connected component containing Id_M.*

If $k \neq \dim(M) + 1$, then $\mathrm{Diff}^k_0(M)$ is a simple group.

Normal subgroups: Enriques conjecture (1894)

Theorem [with Stéphane Lamy].— *Let* $\mathbf{k}$ *be an algebraically closed field.*
The Cremona group $\mathrm{Cr}_2(\mathbf{k})$ *is not simple.*

Theorem [Dahmani, Guirardel, Osin].— *Let* $\mathbf{k}$ *be an algebraically closed field. The Cremona group* $\mathrm{Cr}_2(\mathbf{k})$ *is subquotient universal: Every countable group embeds in a quotient of* $\mathrm{Cr}_2(\mathbf{k})$*.*

Theorem .— *The Cremona group* $\mathrm{Cr}_2(\mathbf{k})$ *satisfies Tits alternative: If* Γ *is a finitely generated subgroup of* $\mathrm{Cr}_2(\mathbf{k})$*, then* Γ *contains a free non abelian subgroup or a finite index solvable subgroup.*

- Tits alternative is satisfied in $\mathrm{GL}_n(\mathbf{k})$;
- Tits alternative is **not** satisfied in $\mathrm{Diff}^\infty(\mathbb{S}^1)$.

Open Problem:
Does $\mathrm{Cr}_n(\mathbf{C})$ *satisfy Tits alternative for* $n \geq 3$ *?*

— III —

Degree Growth and Geometry

- For $f \in \mathrm{Cr}_2(\mathbf{k})$, the **asymptotic degree** is

$$\lambda(f) := \lim_{n \to +\infty} (\deg(f^n)^{1/n}).$$

  If $\lambda(f) > 1$, it is a **Salem or Pisot number**.

- $h(X, Y) = (Y + X^3, X)$, $\deg(h^n) = 3^n$, and $\lambda(h) = 3$.
- $g(X, Y) = (X^2 Y, XY)$, $\lambda(g) = (3 + \sqrt{5})/2$.
- $f(X, Y) = (X + 1, XY)$, $\deg(f^n) = n + 1$ because

$$f^n(X, Y) = (X + n, X(X + 1) \cdots (X + n - 1) Y)$$

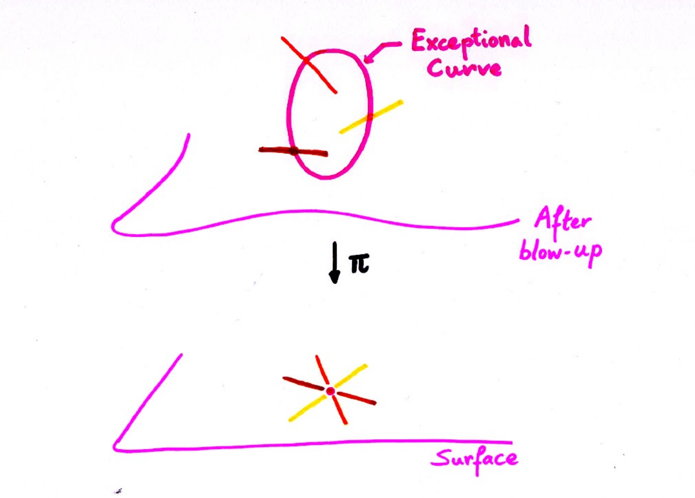
Theorem [Gizatullin, C., Diller-Favre].— *Let f be an element of $\mathrm{Cr}_2(\mathbf{k})$. If $\lambda(f) = 1$ then one of the following is satisfied*

- *$deg(f^n)$ is bounded, and there exists $m > 0$ such that f^m is conjugate to an element of $\mathrm{PGL}_3(\mathbf{k})$;*
- *$deg(f^n) \simeq c^{ste} n$, and f preserves a pencil of rational curves* **(De Jonquières twists)**;
- *$deg(f^n) \simeq c^{ste} n^2$, and f preserves a pencil of curves of genus 1* **(Halphen twists)**.

— IV —

Blow-ups
and
infinite dimensional Hyperbolic space

Blow-up of a point q



- Second homology group

$$H_2(\mathbb{P}^2(\mathbf{C}), \mathbf{Z}) = \mathbf{Z}\,\mathbf{e}_0$$

where

$$\mathbf{e}_0 = \text{class of a line } \ H \simeq \mathbb{P}^1(\mathbf{C}) \subset \mathbb{P}^2(\mathbf{C}).$$

- Blow-up a point p_1 : a new curve $E_1 \simeq \mathbb{P}^1(\mathbf{C})$ appears.

$$H_2(X_1, \mathbf{Z}) = \mathbf{Z}\,\mathbf{e}_0 \oplus \mathbf{Z}\,\mathbf{e}_1, \text{ with } \mathbf{e}_0^2 = 1, \mathbf{e}_0 \cdot \mathbf{e}_1 = 0, \mathbf{e}_1^2 = -1$$

- Blow-up n points p_i, we get

$$H_2(X_n, \mathbf{Z}) = \mathbf{Z}\,\mathbf{e}_0 \oplus \mathbf{Z}\,\mathbf{e}_1 \oplus \ldots \oplus \mathbf{Z}\,\mathbf{e}_n$$

with **intersection form** of signature $(1, n)$.

Inductive Limit and Hyperbolic Space

- Blow-up **all** points successively

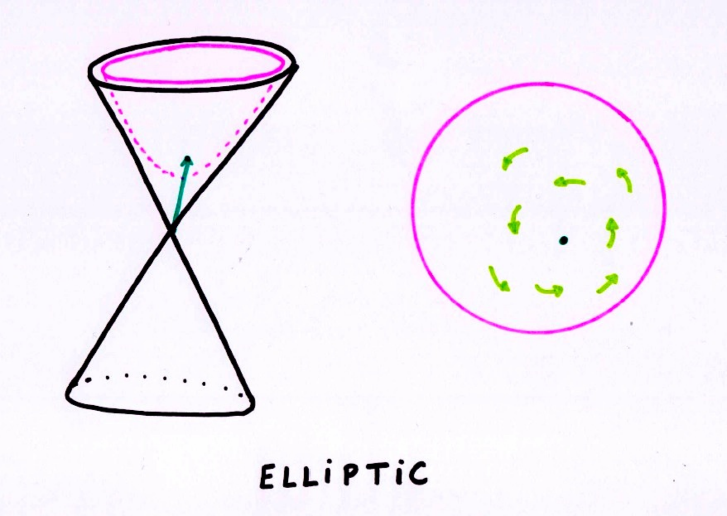
$$\mathcal{Z} = \text{ injective limit of (co)homology groups } H_2(X_i, \mathbf{Z}).$$

The intersection form has signature $(1, \infty)$.

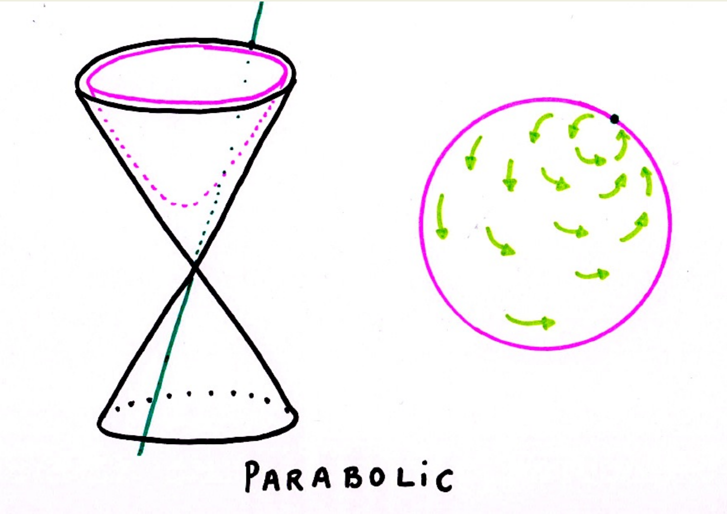
Fact [Yuri Manin's Remark].— *The Cremona group* $\mathrm{Cr}_2(\mathbf{C})$ *acts faithfully by isometries on* $\mathcal{Z}$*.*

- **Hyperbolic space** $\mathbb{H}^\infty$:
  - $\mathbb{H}^\infty_0 = \{u \in \mathcal{Z} \otimes \mathbf{R} \,|\, u^2 = 1 \text{ and } u \cdot \mathbf{e}_0 > 0\}$
  - metric : $\cosh(\mathrm{dist}(u, v)) = u \cdot v$
  - complete $\mathbb{H}^\infty_0$ with respect to this metric to get $\mathbb{H}^\infty$.

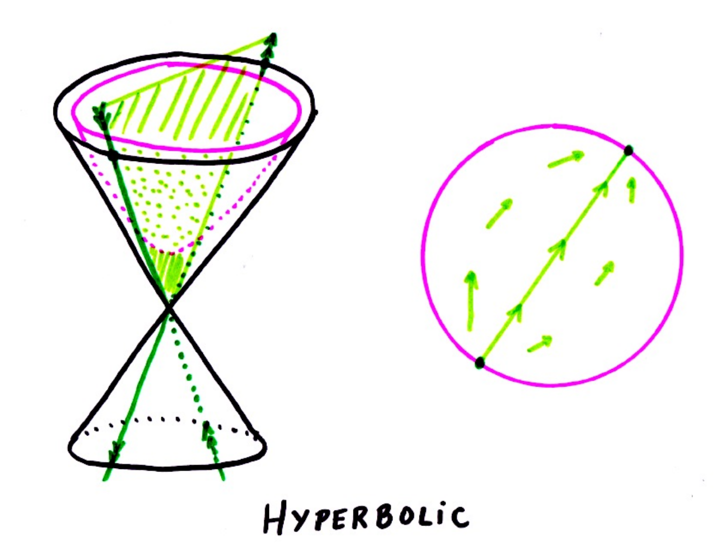
Elliptic Isometries = Bounded degrees



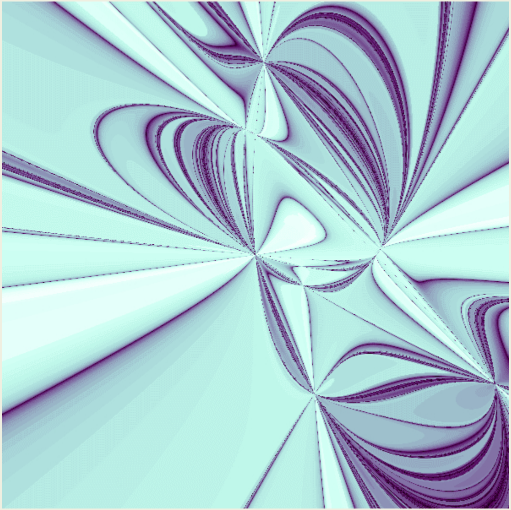
Parabolic Isometries = Halphen and De Jonquières Twists



Loxodromic Isometries = Chaotic Dynamics = $\lambda(f) > 1$

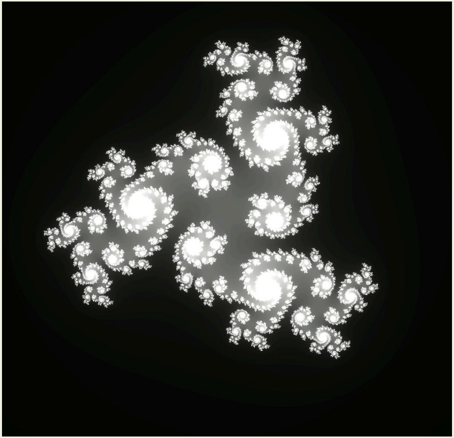


Loxodromic Isometries = Chaotic Dynamics = $\lambda(f) > 1$



(picture by C. McMullen)

- $h : (X, Y) \mapsto (Y + X^3 + c, aX)$



$c = 0.52 + 0.46\sqrt{-1}$