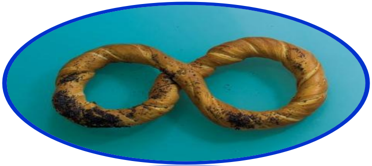
KAM theory: a journey from conservative to dissipative systems

Alessandra Celletti

Department of Mathematics
University of Roma "Tor Vergata"

4 July 2012



Outline

Outline

1. Introduction

• Kolmogorov–Arnold–Moser (KAM) theory provides results on quasi–periodic motions in non–integrable dynamical systems and in particular on the persistence of invariant tori in nearly–integrable Hamiltonian systems.

• We present a recent extension of the theory to dissipative systems (conformally symplectic).

• KAM theory gives also a powerful tool to compute explicit estimates.

• Problem: show that estimates are consistent with experimental values.

• The original formulations gave results far from reality, but computer–assisted proofs allow to obtain results in agreement with the experimental values.

• Aim of the talk: to present the theoretical (conservative and dissipative) results and to show some effective applications to model problems.

• At the ICM in 1954 A.N. Kolmogorov gave the closing lecture titled *"The general theory of dynamical systems and classical mechanics"* on the persistence of quasi–periodic motions under small perturbations of an integrable system. V.I. Arnold (1963) used a different approach and generalized to Hamiltonian systems with degeneracies, while J.K. Moser (1962) covered the finitely differentiable case.

• The theory can be developed under two main assumptions:

- the frequency of motion must obey a Diophantine condition (to get rid of the classical small divisor problem);
- a non–degeneracy condition must be satisfied (to ensure the solution of the cohomological equations providing the approximate solutions).

• KAM theory was motivated by stability problems in Celestial Mechanics, following the works of Laplace, Lagrange, Poincaré, etc.

• KAM theory applies to *nearly–integrable* systems of the form

$$\mathcal{H}(y,x) = h(y) + \varepsilon f(y,x) \ ,$$

where $y \in \mathbb{R}^n$ (actions), $x \in \mathbb{T}^n$ (angles), $\varepsilon > 0$ is a small parameter.

• In the *integrable* approximation $\varepsilon = 0$ Hamilton's equations are solved as

$$\begin{aligned} \dot{y} &= -\frac{\partial h(y)}{\partial x} = 0 \qquad \Rightarrow \qquad y(t) = y(0) = const. \\ \dot{x} &= \frac{\partial h(y)}{\partial y} \equiv \omega(y) \qquad \Rightarrow \qquad x(t) = \omega\left(y(0)\right) t + x(0) \ , \end{aligned}$$

where $(y(0), x(0))$ are the initial conditions. The solution takes place on a torus with frequency $\omega = \omega(y(0))$ and we look for its persistence as $\varepsilon \neq 0$.

• We shall consider also *nearly–integrable dissipative* systems with dissipative constant $\lambda > 0$ and drift term μ:

$$\begin{aligned} \dot{y} &= -\varepsilon \frac{\partial f(y,x)}{\partial x} - \lambda(y - \mu), \\ \dot{x} &= \omega(y) + \varepsilon \frac{\partial f(y,x)}{\partial y} \ . \end{aligned}$$

• An application to the N–body problem in Celestial Mechanics was given by Arnold, who proved the existence of a positive measure set of initial data providing quasi–periodic tori for nearly circular and nearly coplanar orbits.

• Quantitative estimates on a three–body model were given by M. Hénon, based on the original versions; the results were far from reality (at best for primaries mass-ratio 10^{-48} vs. Jupiter-Sun 10^{-3}) and Hénon concluded: *"Ainsi, ces théorèmes, bien que d'un très grand intérêt théorique, ne semblent pas pouvoir en leur état actuel être appliqués á des problèmes pratiques".*

• A challenge came with the *computer–assisted* proofs, where rounding–off and propagation errors are controlled through *interval arithmetic*. One obtains KAM results comparable with the physical (or numerical) expectation.

• Dissipative effects are often non negligible; a dissipative - conformally symplectic - KAM theory (R. Calleja, A.C., R. de la Llave, 2011) shows the existence of quasi–periodic attractors without requiring near–integrability or action–angle variables. It provides an efficient numerical technique to determine the breakdown threshold and very refined quantitative estimates.

Outline

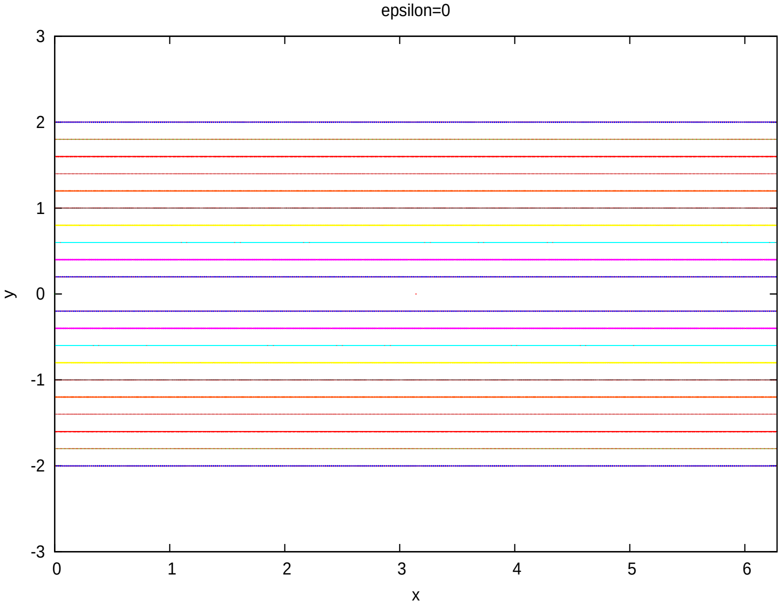
Conservative Standard Map

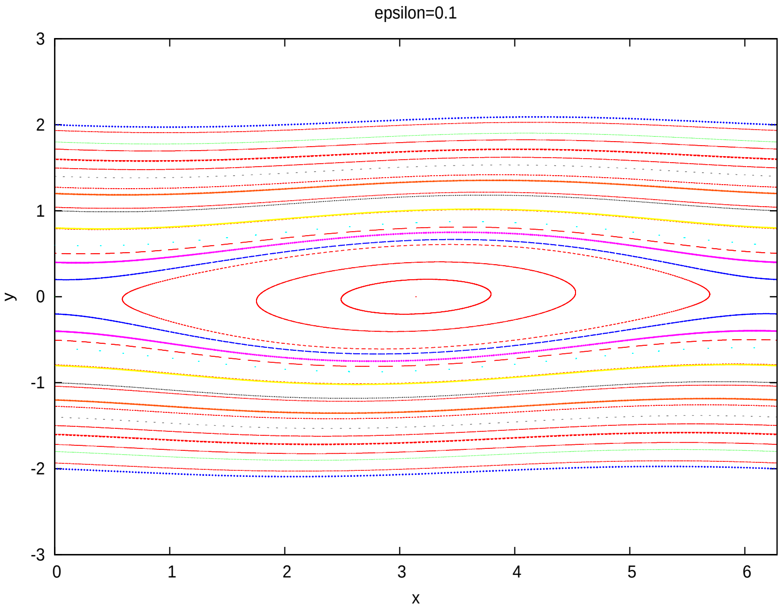
It is described by the equations

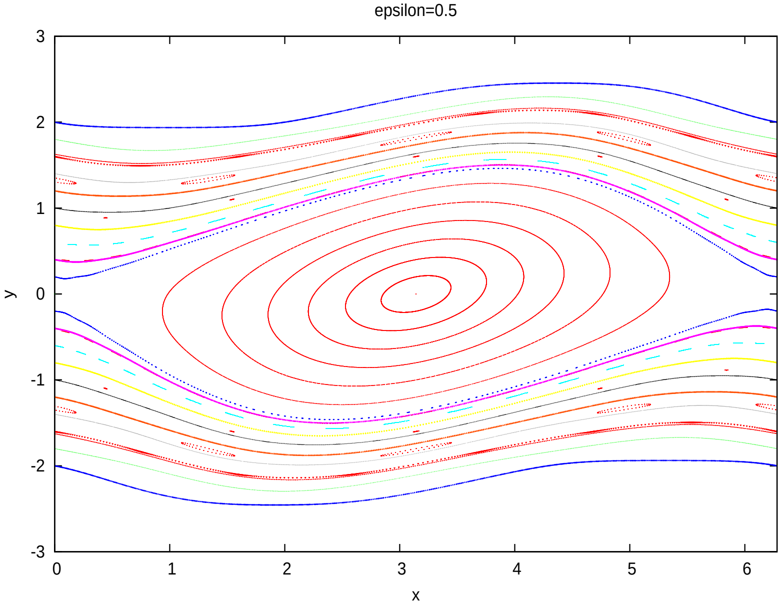
$$\begin{aligned} y' &= y + \varepsilon\, g(x) \qquad\qquad y \in \mathbb{R}\,,\ x \in \mathbb{T} \\ x' &= x + y'\,, \end{aligned}$$

with $\varepsilon > 0$ *perturbing parameter*, $g = g(x)$ analytic function.
• Classical (Chirikov) standard map: $g(x) = \sin x$.

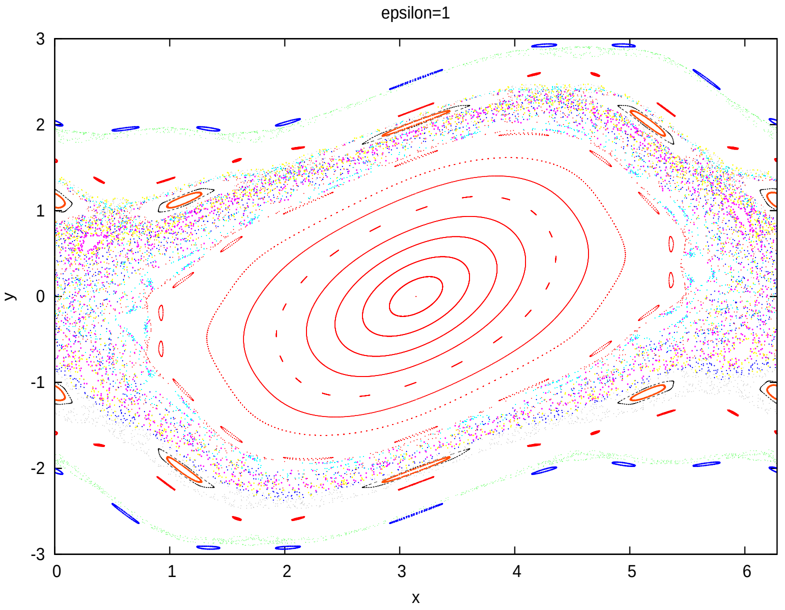
• SM is integrable for $\varepsilon = 0$, non–integrable for $\varepsilon \neq 0$.
• KAM theory provides the existence of invariant curves run with quasi–periodic motions.
• The existence of 2 KAM curves provides a strong stability property in the sense of confinement in phase space between bounding invariant tori.


epsilon=0
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x


epsilon=0.1
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x



epsilon=0.5
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x


epsilon=1
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x

Dissipative Standard Map:

It is described by the equations

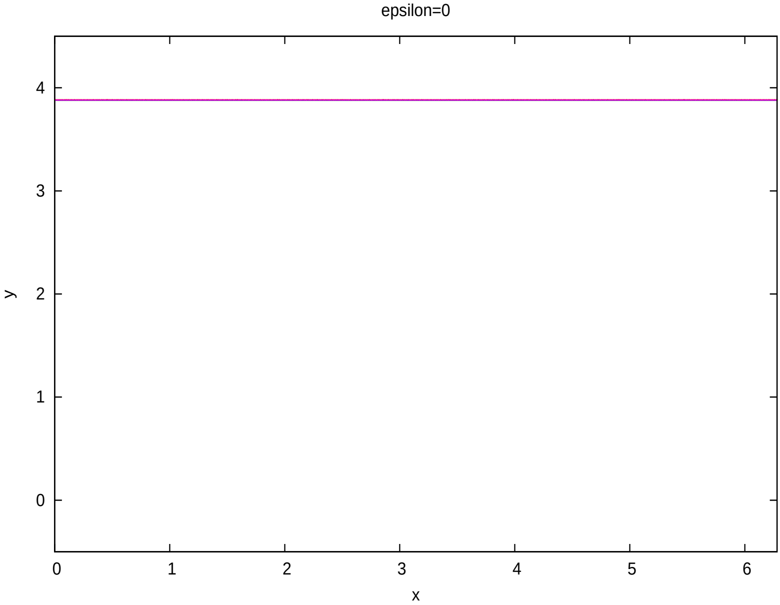
$$\begin{aligned} y' &= \lambda y + \mu + \varepsilon\, g(x) \qquad\qquad && y \in \mathbb{R}\,,\ x \in \mathbb{T} \\ x' &= x + y'\,, && \lambda, \mu, \varepsilon \in \mathbb{R}_+\,, \end{aligned}$$

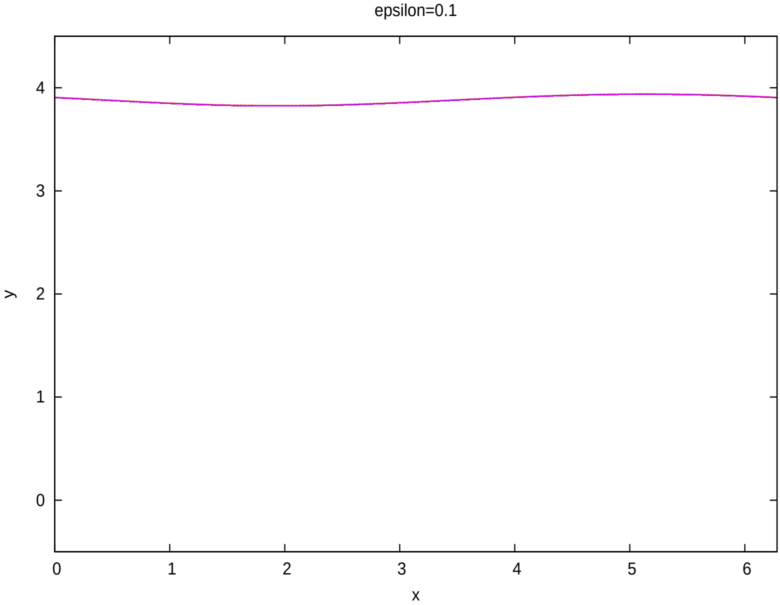
$0 < \lambda < 1$ dissipative parameter, μ = drift parameter.

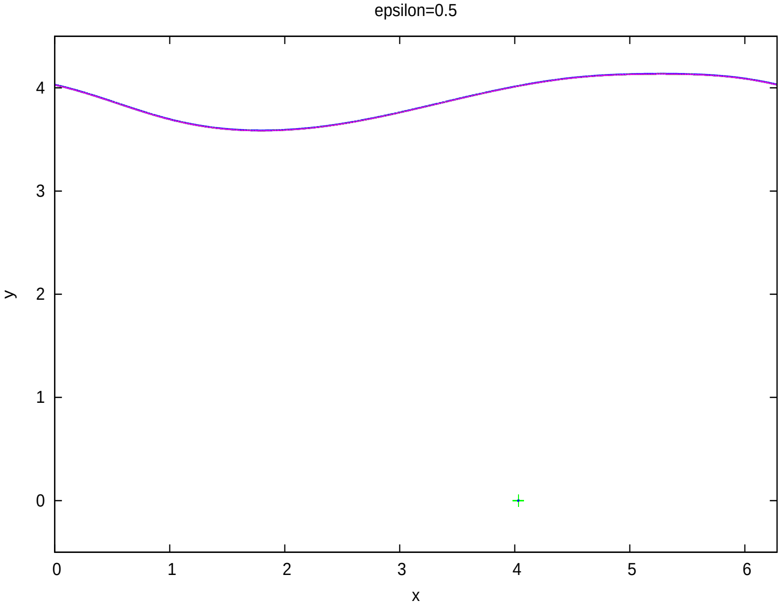
• $\lambda = 1$, $\mu = 0$ conservative SM.
• For $\varepsilon = 0$ the trajectory $\{y = \omega \equiv \frac{\mu}{1-\lambda}\} \times \mathbb{T}$ is invariant:

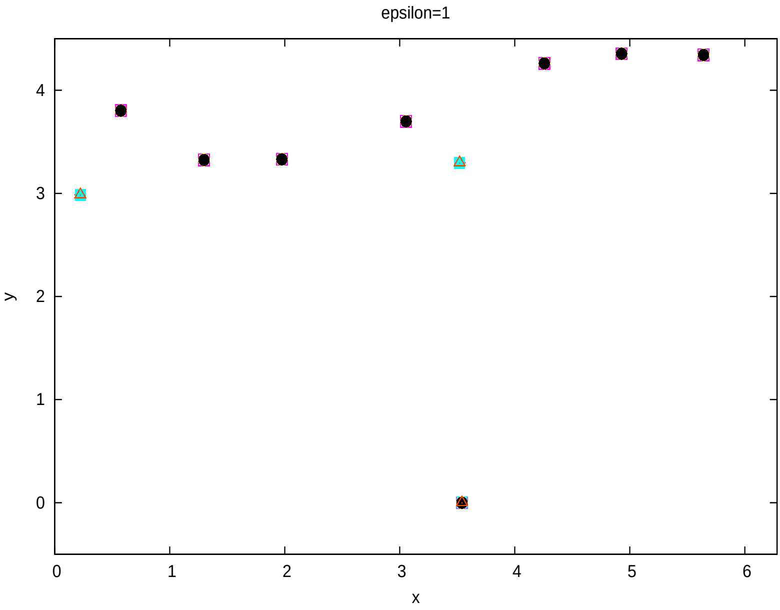
$$y' = y = \lambda y + \mu\,, \quad \omega = y \quad \to \quad \omega = \lambda\omega + \mu \quad \to \quad \omega \equiv \frac{\mu}{1-\lambda}\,.$$

• Invariant attractors are determined by solving an invariance equation and by looking for a suitable drift parameter, since in dissipative systems one cannot adjust the frequency by changing the initial conditions.


epsilon=0
y
4
3
2
1
0
0
1
2
3
4
5
6
x


epsilon=0.1
y
4
3
2
1
0
x
0
1
2
3
4
5
6


epsilon=0.5
4
3
2
1
0
y
0
1
2
3
4
5
6
x


epsilon=1
y
x
0
1
2
3
4
5
6
0
1
2
3
4

Outline

Conservative and conformally symplectic KAM theorems

• Let $\mathcal{M} = U \times \mathbb{T}^n$ be the phase space with $U \subseteq \mathbb{R}^n$ open, simply connected domain with a smooth boundary; let $\mathcal{M}$ be endowed with the standard scalar product and a symplectic form Ω.

Definition

A diffeomorphism f on $\mathcal{M}$ is *conformally symplectic*, if there exists a function $\lambda : \mathcal{M} \to \mathbb{R}$ such that (f^* denotes the pull–back via f)

$$f^* \Omega = \lambda \Omega \, .$$

• For $n = 1$ any diffeomorphism is conformally symplectic with λ depending on the coordinates; $\lambda =$constant for $n \geq 2$; $\lambda = 1$ in the *symplectic* case.

Definition

We say that a vector field X is a *conformally symplectic flow* if, denoting by L_X the Lie derivative, there exists a function $\lambda : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$L_X \Omega = \lambda \Omega \, .$$

• We denote by $\omega \in \mathbb{R}^n$ the frequency vector, which satisfies the Diophantine condition in the case of diffeomorphisms

$$|\frac{\omega \cdot q}{2\pi} - p|^{-1} \le C|q|^{\tau} , \qquad p \in \mathbb{Z} , \quad q \in \mathbb{Z}^n \backslash \{0\} ,$$

while in the case of flows we assume

$$|\omega \cdot q|^{-1} \le C|q|^{\tau} , \qquad q \in \mathbb{Z}^n \backslash \{0\} ,$$

for $C > 0$, $\tau > 0$. We denote by $D(C, \tau)$ the corresponding set of Diophantine vectors, which is of full Lebesgue measure in $\mathbb{R}^n$.

Definition

Let $\mathcal{M} \subseteq \mathbb{R}^n \times \mathbb{T}^n$ be a symplectic manifold and let $f : \mathcal{M} \to \mathcal{M}$ be a symplectic map. A *KAM surface* with frequency $\omega \in D(C, \tau)$ is an n–dimensional invariant surface described parametrically by an embedding $K : \mathbb{T}^n \to \mathcal{M}$, which is the solution of the invariance equation:

$$f \circ K(\theta) = K(\theta + \omega) \ . \tag{1}$$

For a family f_μ of conformally symplectic diffeomorphisms depending on a real parameter μ, look for $\mu = \mu_*$ and an embedding K, such that

$$f_{\mu_*} \circ K(\theta) = K(\theta + \omega) \ .$$

For conformally symplectic vector fields X_μ look for μ_* and K, such that

$$X_{\mu_*} \circ K(\theta) = (\omega \cdot \partial_\theta) \, K(\theta) \ .$$

- Invariant tori are Lagrangian; if f is confor. symplectic, $|\lambda| \neq 1$ and K satisfies (1):

$$K^* \Omega = 0 \ . \tag{2}$$

If f is symplectic and ω is irrational, then the torus is Lagrangian.

• **Main idea:** to prove the existence of a KAM surface, try to solve the invariance equation, starting with an approximate solution satisfying the invariance equation up to an error term.

• Under non–degeneracy conditions, perform a Newton's step by adapting a system of coordinates near solutions of the invariance equation.

• A key argument is that in the neighborhood of an invariant torus, there exists an explicit change of coordinates so that the linearization of the invariance equation is transformed into a constant coefficient equation.

• Using Nash–Moser theory one can construct a sequence of approximate solutions, defined on a suitable scale of Banach spaces, through a quadratically convergent method.

• Estimates on the norms of the functions involved show that such iteration converges quadratically, if the norm of the initial error is sufficiently small.

• Estimates are given on the analytic and Sobolev spaces.

Definition

Analytic norm. Given $\rho > 0$, we define $\mathbb{T}^n_\rho$ as the set

$$\mathbb{T}^n_\rho = \{\theta \in \mathbb{C}^n/(2\pi\mathbb{Z})^n : \ \mathrm{Re}(\theta) \in \mathbb{T}^n, \ |\mathrm{Im}(\theta_j)| \leq \rho \ , \ \ j = 1, ..., n\} \ ;$$

we denote by $\mathcal{A}_\rho$ the set of analytic functions in $Int(\mathbb{T}^n_\rho)$ with the norm

$$\|f\|_\rho = \sup_{\theta \in \mathbb{T}^n_\rho} |f(\theta)| \ .$$

Definition

Sobolev norm. Expand in Fourier series $f(z) = \sum_{k \in \mathbb{Z}^n} \widehat{f}_k e^{2\pi ikz}$ and for $m > 0$:

$$H^m = \left\{ f : \mathbb{T}^n \to \mathbb{C} \ : \ \|f\|^2_m \equiv \sum_{k \in \mathbb{Z}^n} |\widehat{f}_k|^2 (1 + |k|^2)^m < \infty \right\} .$$

• Advantages of Sobolev norms: they apply to mappings with finite regularity and it provides an efficient numerical technique for the breakdown threshold.
• $J = J(x)$ is the matrix representing Ω at x: $\forall$ vectors u, v, $\Omega_x(u, v) = (u, J(x)v)$.

Theorem (conservative case, R. de la Llave et al.)

Let $\omega \in D(C,\tau)$ and let $f : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n$ symplectic, analytic mapping. Assume K_0 is an approximate solution of (1) with error term $E_0 = E_0(\theta)$. Let $N(\theta) \equiv (DK_0(\theta)^T\, DK_0(\theta))^{-1}$; let $S(\theta)$ be

$$\begin{aligned} S(\theta) \quad &\equiv N(\theta+\omega)^T DK_0(\theta+\omega)^T \Big[Df(K_0(\theta))\, J(K_0(\theta))^{-1} DK_0(\theta)\, N(\theta) \\ &\qquad - J(K_0(\theta+\omega))^{-1}\, DK_0(\theta+\omega)\, N(\theta+\omega)\, A(\theta) \Big] \end{aligned}$$

with $A(\theta) = \mathrm{Id}$. Assume that S satisfies the non–degeneracy condition

$$\det \langle S(\theta) \rangle \neq 0\ ,$$

where $\langle \cdot \rangle$ is the average. Let $0 < \delta < \frac{\rho}{2}$; if the solution is suff. approximate, i.e.

$$\|E_0\|_\rho \leq C_1\, C^{-4}\, \delta^{4\tau} \qquad (C_1 > 0)\ ,$$

then there exists an exact solution $K_e = K_e(\theta)$ of (1), such that

$$\|K_e - K_0\|_{\rho-2\delta} < C_2\, C^2\, \delta^{-2\tau}\, \|E_0\|_\rho \qquad (C_2 > 0)\ .$$

Theorem (conformally sympl. case, R. Calleja, A.C., R. de la Llave)

Let $\omega \in D(C,\tau)$ *and* $f_\mu : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n$ *conformally symplectic. Let* (K_0, μ_0) *be an approximate solution and* E_0 *error term. Let* $M(\theta)$ *be the* $2n \times 2n$ *matrix*

$$M(\theta) = [DK_0(\theta) \mid J(K_0(\theta))^{-1}\, DK_0(\theta)N(\theta)]\ .$$

Assume the following non–degeneracy condition:

$$\det \begin{pmatrix} \langle S \rangle & \langle SB^0 \rangle + \langle \widetilde{A}_1 \rangle \\ (\lambda - 1)\mathrm{Id} & \langle \widetilde{A}_2 \rangle \end{pmatrix} \neq 0\ ,$$

with $A(\theta) = \lambda\,\mathrm{Id}$, $\widetilde{A}_1$, $\widetilde{A}_2$ *first and second n columns of* $\tilde{A} = M^{-1}(\theta + \omega) D_{\mu_0} f_{\mu_0} \circ K_0$, $B^0 = B - \langle B \rangle$ *solution of* $\lambda B^0(\theta) - B^0(\theta + \omega) = -(\widetilde{A}_2)^0(\theta)$. *Let* $0 < \delta < \frac{\rho}{2}$*; if the solution is suff. approximate, i.e.*

$$\|E_0\|_\rho \leq C_3\, C^{-4}\, \delta^{4\tau} \qquad (C_3 > 0)\ ,$$

there exists an exact solution (K_e, μ_e)*, such that*

$$\|K_e - K_0\|_{\rho - 2\delta} \leq C_4\, C^2\, \delta^{-2\tau}\, \|E_0\|_\rho\ , \quad |\mu_e - \mu_0| \leq C_5\, \|E_0\|_\rho \quad (C_4, C_5 > 0)\ .$$

- A remark on the non–degeneracy conditions.
- For the conservative standard map

$$\begin{aligned} y' &= y + \varepsilon\, g(x) \\ x' &= x + y' \,, \end{aligned}$$

the non–degeneracy is equivalent to the *twist* condition:

$$\frac{\partial x'}{\partial y} \neq 0 \,,$$

namely the lift transforms any vertical line always on the same side.

- For the (generalized) dissipative standard map

$$\begin{aligned} y' &= \lambda y + s(\mu) + \varepsilon\, g(x) \\ x' &= x + y' \,, \end{aligned}$$

the non–degeneracy condition involves the twist condition and that $\frac{ds(\mu)}{d\mu} \neq 0$ which corresponds to a non–degeneracy w.r.t. to the parameters.

Outline

Sketch of the Proof

Step 1: approximate solution and linearization
Step 2: determine the new approximation
Step 3: solve the cohomological equation
Step 4: convergence of the iterative step
Step 5: local uniqueness

• Analytic tools:

- exponential decay of Fourier coefficients of analytic functions;
- Cauchy estimates to bound derivatives of analytic functions in smaller domains;
- quantitative analysis of the cohomology equations;
- abstract implicit function theorem.

Step 1: approximate solution and linearization

- Let (K,μ) be an approximate solution. In coordinates $K^*\Omega=0$ becomes $DK^T(\theta)\, J\circ K(\theta)\, DK(\theta)=0$, showing that the tangent space is $\text{Range}\Big(DK(\theta)\Big)\oplus\text{Range}\Big(J^{-1}\circ K(\theta)DK(\theta)\Big)$.

- Define $M(\theta)=[DK(\theta)\mid J^{-1}\circ K(\theta)\, DK(\theta)N(\theta)]$; one can show that up to a remainder R:

$$Df_\mu\circ K(\theta)\, M(\theta)=M(\theta+\omega)\begin{pmatrix}\text{Id} & S(\theta)\\ 0 & \lambda\text{Id}\end{pmatrix}+R(\theta)\,. \qquad (R)$$

- Find $K'=K+MW$, $\mu'=\mu+\sigma$ satisfying

$$f_{\mu'}\circ K'(\theta)-K'(\theta+\omega)=E'(\theta) \qquad (APPR-INV)'$$

where E' is quadratically smaller provided

$$Df_\mu\circ K(\theta)\, M(\theta)W(\theta)-M(\theta+\omega)\, W(\theta+\omega)+D_\mu f_\mu\circ K(\theta)\sigma=-E(\theta)\,.$$

Step 2: determine the new approximation

• Using (R) and neglecting h.o.t., one obtains the cohomological equations with constant coefficients for $W=(W_1,W_2)$, σ for known S, $\widetilde{E}\equiv(\widetilde{E}_1,\widetilde{E}_2)$, $\widetilde{A}\equiv[\widetilde{A}_1|\,\widetilde{A}_2]$:

$$\begin{aligned} W_1(\theta)-W_1(\theta+\omega) &= -\widetilde{E}_1(\theta)-S(\theta)W_2(\theta)-\widetilde{A}_1(\theta)\,\sigma & (A)\\ \lambda W_2(\theta)-W_2(\theta+\omega) &= -\widetilde{E}_2(\theta)-\widetilde{A}_2(\theta)\,\sigma & (B)\end{aligned}$$

• (A) involves small (zero) divisors, since for $k=0$ one has $1-e^{ik\cdot\omega}=0$ in

$$W_1(\theta)-W_1(\theta+\omega)=\sum_k \widehat{W}_{1,k}\,e^{ik\cdot\theta}(1-e^{ik\cdot\omega})\ .$$

• (B) always solvable for any $|\lambda|\neq 1$ by a contraction mapping argument.

• Non–degeneracy condition: computing the averages of eqs. (A), (B), determine $\langle W_2\rangle$, σ by solving $(W_2=\langle W_2\rangle+B^0+\sigma\tilde{B}^0)$

$$\begin{pmatrix} \langle S\rangle & \langle SB^0\rangle+\langle\widetilde{A}_1\rangle \\ (\lambda-1)\mathrm{Id} & \langle\widetilde{A}_2\rangle \end{pmatrix}\begin{pmatrix}\langle W_2\rangle\\ \sigma\end{pmatrix}=\begin{pmatrix}-\langle S\tilde{B}^0\rangle-\langle\widetilde{E}_1\rangle\\ -\langle\widetilde{E}_2\rangle\end{pmatrix}\ .$$

Step 3: solve the cohomological equation

• The non–average parts of W_1, W_2 are obtained by solving cohomological equations of the form

$$\lambda\varphi(\theta) - \varphi(\theta+\omega) = \eta(\theta) \ ,$$

for $\varphi : \mathbb{T}^n \to \mathbb{C}$, $\eta : \mathbb{T}^n \to \mathbb{C}$, where $\lambda \in \mathbb{C}$, $\omega \in \mathbb{R}^n$ are given.

Lemma

Let $|\lambda| \in [A, A^{-1}]$ *for* $0 < A < 1$, $\omega \in D(C,\tau)$, $\eta \in \mathcal{A}_\rho$, $\rho > 0$ *or* $\eta \in H^m$, $m \geq \tau$, *and*

$$\int_{\mathbb{T}^n} \eta(\theta)\, d\theta = 0 \ .$$

Then, there is one and only one solution φ *with zero average and if* $\varphi \in \mathcal{A}_{\rho-\delta}$ *for every* $\delta > 0$ *or* $\varphi \in H^{m-\tau}$*:*

$$\begin{aligned} \|\varphi\|_{\mathcal{A}_{\rho-\delta}} &\leq C_6\, C\, \delta^{-\tau} \|\eta\|_{\mathcal{A}_\rho} \ , \\ \|\varphi\|_{H^{m-\tau}} &\leq C_7\, C\, \|\eta\|_{H^m} \ . \end{aligned}$$

Step 4: convergence of the iterative step

- The invariance equation is satisfied with an error quadratically smaller, i.e.

$$\|E'\|_{\mathcal{A}_{\rho-\delta}} \le C_8 \delta^{-2\tau} \|E\|^2_{\mathcal{A}_\rho} , \qquad \|E'\|_{H^{m-\tau}} \le C_9 \|E\|^2_{H^m} ,$$

whenever $f_{\mu+\sigma} \circ (K+W)(\theta) - (K+W)(\theta+\omega) = E'(\theta)$.

- The procedure can be iterated to get a sequence of approximate solutions, say $\{K_j\}$ or $\{K_j, \mu_j\}$.

- Convergence is obtained through an *abstract implicit function theorem*, alternating the iteration with carefully chosen smoothings operators defined in a scale of Banach spaces (analytic functions or Sobolev spaces)

- An analytic smoothing shows the convergence of the iterative step to the exact solution: the sequence of approximate solutions is constructed in smaller analyticity domains, but the loss of analyticity domain slows down, so that the exact solution is defined with a positive radius of analyticity.

Step 5: local uniqueness

• A local uniqueness result is proved under smallness conditions by showing that if there exist two solutions K_a, K_b or (K_a, μ_a), (K_b, μ_b), then there exists $s \in \mathbb{R}^n$ such that

$$K_b(\theta) = K_a(\theta + s)$$

in the conservative setting together with

$$\mu_a = \mu_b$$

in the conformally symplectic framework.

Outline

Break–down of quasi–periodic tori/attractors

• Write the embedding as $K(\theta) = (\omega + u(\theta) - u(\theta - \omega), \theta + u(\theta))$, where u satisfies

$$D_1 D_\lambda u(\theta) - \varepsilon \sin(\theta + u(\theta)) + \gamma = 0\,, \qquad \gamma = \omega(1 - \lambda) - \mu$$

with $D_\lambda u(\theta) = u(\theta + \frac{\omega}{2}) - \lambda u(\theta - \frac{\omega}{2})$, and $\lambda = 1$, $\mu = 0$ in the conservative case.

• Close to breakdown: blow up of the Sobolev norms of a trigonometric approximation

$$u^{(M)}(\vartheta) = \sum_{|k| \leq M} \hat{u}_k\, e^{ik\vartheta}\ .$$

• A regular behavior of $\|u^{(M)}\|_m$ as ε increases (for λ fixed) provides evidence of the existence of the invariant attractor. Table showing ε_{crit} for $\omega_r = 2\pi\frac{\sqrt{5}-1}{2}$.

Conservative case	Dissipative case	
ε_{crit}	λ	ε_{crit}
0.9716	0.9	0.9721
	0.5	0.9792

Greene's method, periodic orbits and Arnold's tongues

• Conservative standard map. Greene's method: the breakdown of an invariant curve with frequency ω is strictly related to the stability character of the approximating periodic orbits with periods $\frac{p_j}{q_j} \to \omega$.

• Dissipative standard map. For fixed values of the parameters there is a whole interval of the drift, which admits a periodic orbit with given period: *Arnold tongue*, where one needs to select a periodic orbit.

• Let $\varepsilon_{p_j,q_j}^{\omega_r}$ be the maximal value of ε for which the periodic orbit has a stability transition; the sequence converges to the breakdown threshold of $\omega_r = 2\pi\frac{\sqrt{5}-1}{2}$.

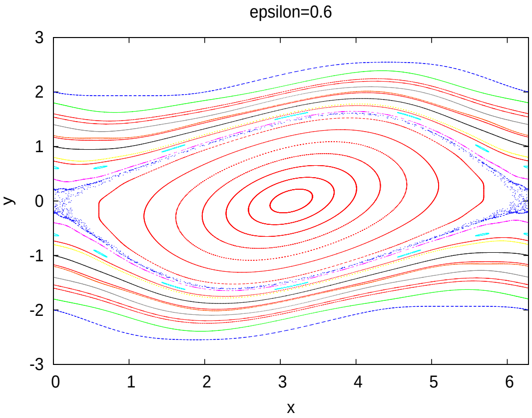
• Method of the periodic orbits (Greene's method):

p_j/q_j	$\varepsilon^{\omega_r}_{p_j,q_j}(cons)$ $\varepsilon_{Sob}=[0.9716]$	$\varepsilon^{\omega_r}_{p_j,q_j}(\lambda=0.9)$ $\varepsilon_{Sob}=[0.972]$	$\varepsilon^{\omega_r}_{p_j,q_j}(\lambda=0.5)$ $\varepsilon_{Sob}=[0.979]$
1/2	0.9999	0.999	0.999
2/3	0.9582	0.999	0.999
3/5	0.9778	0.999	0.999
5/8	0.9690	0.993	0.992
8/13	0.9726	0.981	0.987
13/21	0.9711	0.980	0.983
21/34	0.9717	0.976	0.980
34/55	0.9715	0.975	0.979
55/89	0.9716	0.974	0.979

Outline

Applications

- Standard map
- Rotational dynamics: spin–orbit problem
- Orbital dynamics: three–body problem



Applications

- Standard map
- Rotational dynamics: spin–orbit problem
- Orbital dynamics: three–body problem

Applications

- $\boxed{\text{Standard map}}$
- Rotational dynamics: $\boxed{\text{spin–orbit problem}}$
- Orbital dynamics: $\boxed{\text{three–body problem}}$

Conservative standard map

- Write the embedding as

$$K(\theta) = (\omega + u(\theta) - u(\theta - \omega), \theta + u(\theta)) \ , \qquad \theta \in \mathbb{T} \ ,$$

where $\theta' = \theta + \omega$, u analytic function depending on ε and such that $1 + \frac{\partial u(\theta)}{\partial \theta} \neq 0$. Then u must satisfy

$$D^2 u(\theta) = \varepsilon \sin(\theta + u(\theta))$$

with $Du(\theta) = u(\theta + \frac{\omega}{2}) - u(\theta - \frac{\omega}{2})$.

- The initial approximation is obtained as the finite truncation up to order N:

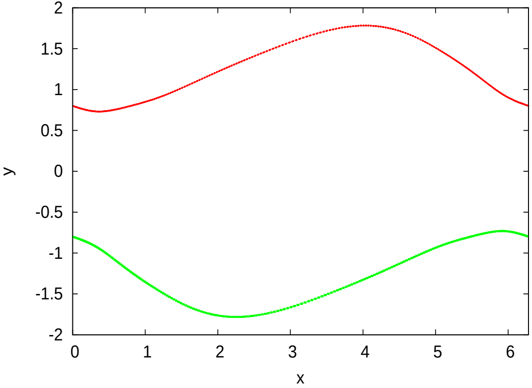
$$u^{(N)}(\theta) = \sum_{j=1}^{N} u_j(\theta) \varepsilon^j \ ,$$

where the functions $u_j = u_j(\theta)$ can be explicitly computed recursively.

Proposition [standard map, A.C., L. Chierchia (1995)]

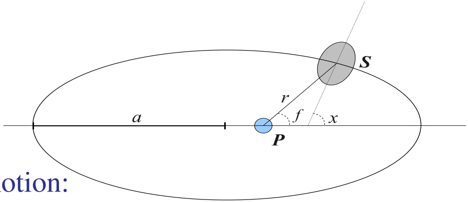
Let $\omega = 2\pi \frac{\sqrt{5}-1}{2}$ and $N = 190$. Then, for any $|\varepsilon| \leq 0.838$ there exists an analytic solution $u = u(\theta)$ defined on $\mathbb{T}^1_\rho \times \{\varepsilon : \ |\varepsilon| \leq 0.838\}$ with $\rho = 5.07 \cdot 10^{-3}$.

• The results is 86% of Greene's value and was improved by R. de la Llave and D. Rana up to 93%, using accurate strategies and efficient computer–assisted algorithms.



Conservative spin–orbit problem

• Spin–orbit problem: triaxial satellite $\mathcal{S}$ (with $A < B < C$) moving on a Keplerian orbit around a central planet $\mathcal{P}$, assuming that the spin–axis is perpendicular to the orbit plane and coincides with the shortest physical axis (all other gravitational and dissipative forces are neglected).



• Equation of motion:

$$\ddot{x} + \varepsilon\left(\frac{a}{r}\right)^3 \sin(2x - 2f) = 0\ , \qquad \varepsilon = \frac{3}{2}\frac{B-A}{C}\ .$$

corresponding to a 1–dim, time–dependent Hamiltonian (KAM tori confine the motion):

$$\mathcal{H}(y, x, t) = \frac{y^2}{2} - \frac{\varepsilon}{2}\left(\frac{a}{r(t)}\right)^3 \cos(2x - 2f(t))\ .$$

• The (Diophantine) frequencies of the bounding tori are

$$\omega_- \equiv 1 - \frac{1}{2 + \frac{\sqrt{5}-1}{2}}\,, \qquad \omega_+ \equiv 1 + \frac{1}{2 + \frac{\sqrt{5}-1}{2}}\,.$$

• Parameterize the tori by $K(\theta, t) = (\omega + Du(\theta, t), \theta + u(\theta, t))$ with $\dot{\theta} = \omega_\pm$, where u must satisfy ($D \equiv \omega \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t}$):

$$D^2 u(\theta, t) = -\varepsilon \left(\frac{a}{r(t)} \right)^3 \sin\Big(2\theta + 2u(\theta, t) - 2f(t) \Big)\,.$$

• The initial approximation is the truncation of u up to order $N = 15$.

• We denote by $\varepsilon_{Moon} = 3.45 \cdot 10^{-4}$ the astronomical value of the Moon.

Proposition [spin–orbit model, A.C. (1990)]

Consider the spin–orbit Hamiltonian defined in $U \times \mathbb{T}^2$ with $U \subset \mathbb{R}$ open set. Then, for the true eccentricity of the Moon $e = 0.0549$, there exist bounding invariant tori with frequencies ω_- and ω_+ for any $\varepsilon \leq \varepsilon_{Moon}$.

Conservative three–body problem

• Consider the motion of a small body P_2 under the gravitational influence of two primaries P_1 and P_3 with masses, respectively, $m_1 > m_3$.
• Assume that the mass of P_2 is negligible, so that P_1 and P_3 move on Keplerian orbits about their common barycenter (*restricted* problem).
• Assume that the orbits of the primaries are circular and that all bodies move on the same plane: *planar, circular, restricted three–body problem* (PCR3BP).

• Adopting suitable normalized units and action–angle Delaunay variables $(L, G) \in \mathbb{R}^2$, $(\ell, g) \in \mathbb{T}^2$, we obtain a 2 d.o.f. Hamiltonian function:

$$\mathcal{H}(L, G, \ell, g) = -\frac{1}{2L^2} - G + \varepsilon R(L, G, \ell, g) .$$

• $\varepsilon = \frac{m_3}{m_1}$ primaries' mass ratio.

• Actions: $L=\sqrt{a}$, $G=L\sqrt{1-e^2}$. Angles: the mean anomaly ℓ, $g=\tilde{\omega}-t$ with $\tilde{\omega}$ argument of perihelion.

• For $\varepsilon=0$ the Hamiltonian describes the Keplerian motion.

• The perturbing function $R=R(L,G,\ell,g)$ represents the interaction with P_3 (expand in Fourier–Taylor series and use a trigonometric approximation).

• The Hamiltonian is degenerate, but it satisfies Arnold's isoenergetic non–degeneracy condition, which guarantees the persistence of invariant tori on a fixed energy surface, i.e. setting $h(L,G)=-\frac{1}{2L^2}-G$:

$$\det\begin{pmatrix} h''(L,G) & h'(L,G) \\ h'(L,G)^T & 0 \end{pmatrix} = \frac{3}{L^4} \neq 0 \qquad \text{for all } \ L \neq 0\ .$$

• Concrete example: Sun, Jupiter, asteroid 12 Victoria with $a = 0.449$ (in Jupiter–Sun unit distance) and $e = 0.22$, so that $L_{\rm V} \simeq 0.670$, $G_{\rm V} \simeq 0.654$.
• Select the energy level as

$$E^*_{\rm V} = -\frac{1}{2L^2_{\rm V}} - G_{\rm V} + \varepsilon_J \langle R(L_{\rm V}, G_{\rm V}, \ell, g) \rangle \simeq -1.769 \ ,$$

where $\varepsilon_J \simeq 10^{-3}$ is the observed Jupiter–Sun mass–ratio. On such (3–dim) energy level prove the existence of two (2–dim) trapping tori with frequencies $\omega_\pm$.

Proposition [three–body problem, A.C., L. Chierchia (2007)]

Let $E = E^*_{\rm V}$. Then, for $|\varepsilon| \le 10^{-3}$ the unperturbed tori with frequencies $\omega_\pm$ can be analytically continued into KAM tori for the perturbed system on the energy level $\mathcal{H}^{-1}\big(E^*_{\rm V}\big)$ keeping fixed the ratio of the frequencies.

• Due to the link between a, e and L, G, this result guarantees that a, e remain close to the unperturbed values within an interval of size of order ε.

Dissipative standard map

- Using $K(\theta) = (\omega + u(\theta) - u(\theta - \omega), \theta + u(\theta))$, the invariance equation is

$$D_1 D_\lambda u(\theta) - \varepsilon \sin(\theta + u(\theta)) + \gamma = 0\ , \qquad \gamma = \omega(1-\lambda) - \mu \tag{3}$$

with $D_\lambda u(\theta) = u(\theta + \frac{\omega}{2}) - \lambda u(\theta - \frac{\omega}{2})$.

Proposition [dissipative standard map, R. Calleja, A.C., R. de la Llave (2012)]

Let $\omega = 2\pi \frac{\sqrt{5}-1}{2}$ and $\lambda = 0.9$; then, for $\varepsilon \leq \varepsilon_{KAM}$, there exists a unique solution $u = u(\theta)$ of (3), provided that $\mu = \omega(1-\lambda) + \langle u_\theta\, D_1 D_\lambda u \rangle$.

- The drift μ must be suitably tuned and cannot be chosen independently from ω.

- Preliminary result: conf. symplectic version, careful estimates, continuation method using the Fourier expansion of the initial approximate solution $\Rightarrow$

$$\varepsilon_{KAM} = \boxed{99\% \text{ of the critical breakdown threshold}\ .}$$

Dissipative spin–orbit problem

• Spin–orbit equation with tidal torque is given by

$$\ddot{x} + \varepsilon \left(\frac{a}{r}\right)^3 \sin(2x - 2f) = -\lambda(\dot{x} - \mu) \ , \tag{4}$$

where λ, μ depend on the orbital (e) and physical properties of the satellite.

Proposition [dissipative spin–orbit problem, A.C., L. Chierchia (2009)]

Let $\lambda_0 \in \mathbb{R}_+$, ω Diophantine. There exists $0 < \varepsilon_0 < 1$, such that for any $\varepsilon \in [0, \varepsilon_0]$ and any $\lambda \in [-\lambda_0, \lambda_0]$ there exists a unique function $u = u(\theta, t)$ with $\langle u \rangle = 0$, such that

$$x(t) = \omega\, t + u(\omega t, t)$$

solves the equation of motion with $\mu = \omega\ (1 + \langle u_\theta^2 \rangle)$.

• This states that the drift and the frequency are not independent, and therefore the frequency and e are linked by a specific relation, thus giving an explanation to Mercury's non–synchronous present state.

Outline

Conclusions and perspectives

• Beside having a theoretical interest, KAM theory can be applied to model problems.

• There are several consequences of the method, like

- a numerically efficient criterion for the break–down of the quasi–periodic solutions;
- the bootstrap of regularity (i.e., that all tori which are smooth enough are analytic if the map is analytic);
- a smooth dependence on the parameters, including the limit of zero dissipation;
- the theory can be extended to lower–dimensional tori.

• Forthcoming applications to interesting physical problems: the spin–orbit model or the restricted, elliptic, planar 3–body problem.

References.

- A.N. Kolmogorov, *On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian*, Dokl. Akad. Nauk. SSR **98** (1954)
- V.I. Arnold, *Proof of A.N. Kolmogorov's theorem on the preservation of quasiperiodic motions under small perturbations of the Hamiltonian*, Russian Math. Surveys, **18** (1963)
- J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen II. Math. Phys. Kl. (1962)

- A.C., *Stability and Chaos in Celestial Mechanics*, Springer–Praxis Publishing Ltd. (2010)
- A.C., L. Chierchia, *KAM Stability and Celestial Mechanics*, Memoirs American Mathematical Society **187**, no. 878 (2007)
- R. de la Llave, A. Gonzalez, A. Jorba and J. Villanueva, *KAM theory without action–angle variables*, Nonlinearity **18** (2005)
- R. de la Llave and D. Rana, *Accurate strategies for K.A.M. bounds and their implementation*, In Computer aided proofs in analysis, IMA Vol. Math. Appl. **28** (1991)

- R. Calleja, A.C., *Breakdown of invariant attractors for the dissipative standard map*, CHAOS (2010)
- R. Calleja, A.C., R. de la Llave, *KAM theory for conformally symplectic systems*, Preprint (2011)
- A.C., L. Chierchia, *Quasi–periodic attractors in Celestial Mechanics*, ARMA **191**, (2008)