Some mathematical aspects of water waves

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Two themes

- Particle paths in Stokes waves (regular periodic travelling waves in irrotational flow over a flat bed)

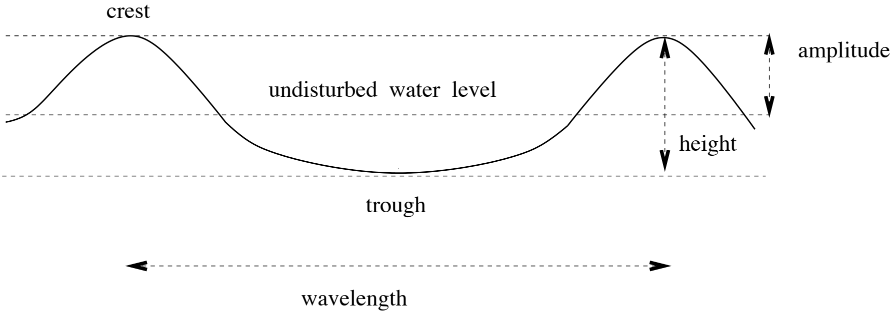
  KEY ASPECT: wave motion $\neq$ water motion (the waves are not moving humps of water but invisible pulses of energy moving through water)

- Wave breaking

  KEY ASPECT: not captured within the framework of linear/weakly nonlinear theory (nonlinearity essential); due to the intractability of the governing equations, current work focuses on the derivation of models that present this feature

1. Particle paths in Stokes waves

Regular wave trains of plane waves (steady periodic two-dimensional waves), termed swell in oceanography



Linear wave theory predicts sinuosoidal wave profiles but in this photograph (taken from the Great Ocean Road, Victoria, Australia) the crest is higher and narrower and the trough broader and less deep.

This wave train in shallow water with a flat bed (photographed near the coast of California) is even further from a sinusoidal wave pattern, being almost flat near the troughs with pronounced elevations near the crests.

The governing equations for travelling gravity water waves

- Water is modelled as a homogeneous incompressible inviscid fluid. Let $\mathbf{u} = (u, v)$ be the fluid velocity. We set $\rho = 1$ for the fluid density and denote by g the gravitational constant of acceleration.
- The equation of mass conservation: $u_x + v_y = 0$
- Euler's equation : $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + (0, -g)$
- Irotational flow: $u_y = v_x$
- The boundary conditions: the kinematic boundary conditions

$$v = \eta_t + u\,\eta_x \quad \text{on the free surface} \quad y = \eta(x,t),$$

$$v = 0 \quad \text{on the flat bed} \quad y = -d,$$

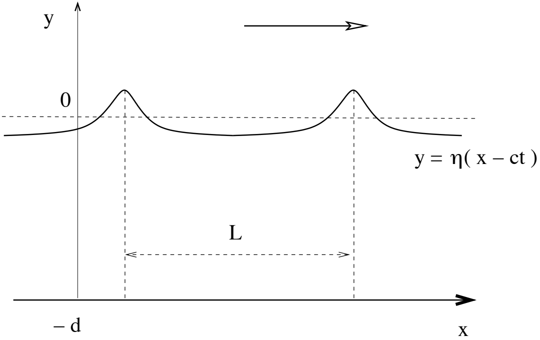
  express the fact that a particle on the boundary is confined to it, and the dynamic boundary condition

$$P = P_{atm} \quad \text{on the free surface} \quad y = \eta(x,t),$$

  decouples the water flow from the motion of the air above it.

Steady periodic waves

Given the wave speed $c > 0$, these are two-dimensional periodic travelling waves: the space-time dependence of the free surface, of the pressure, and of the velocity field has the form $(x - ct)$ and is periodic with period $L > 0$. The wave $y = \eta(x - ct)$ oscillates about the (undisturbed) flat surface $y = 0$: $\int_0^L \eta(x)\, dx = 0$.



Linear theory

Contrary to a possible first impression, what one observes traveling across the sea is not the water but a wave pattern, as enunciated intuitively in the fifteenth century by Leonardo da Vinci: "... the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in its place."

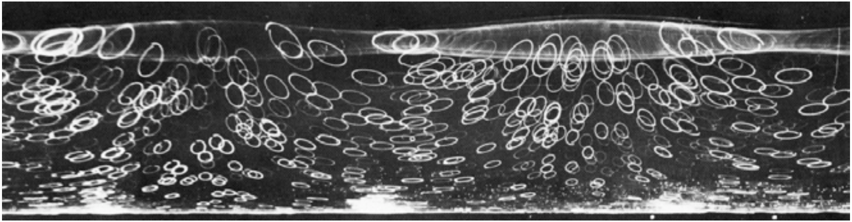
Classical linear theory suggests that the particles perform a circular motion as the wave passes over: individual particles of water apparently do not travel along with the wave, but instead they move in closed orbits. Support for this conclusion seems to be given by some experimental evidence. The Java Applet for Coastal Engineering at

http://www.coastal.udel.edu/faculty/rad/linearplot.html

which shows graphically the presumed orbital particle motions for a given wave height, wave period, and mean water depth.

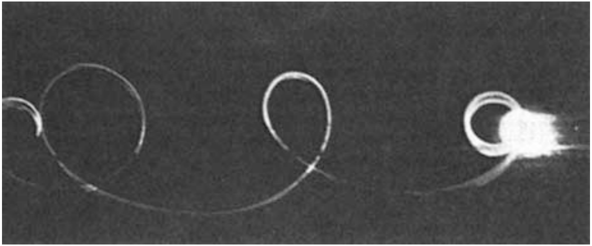
Photograph of particle paths, reproduced from [A. Wallet & F. Ruellan, La Houille Blanche 5 (1950): 483–489]

The water is kept in a narrow glass container with parallel walls, and small light absorbing particles such as metal fillings mixed in. Photographing across the container one visualizes the particle paths by a long exposure (as long as a half period): the eye involuntarily connects the various positions occupied by the particle and one can clearly distinguish elliptical arcs.



Laboratory experiment for tracing the path of a surface particle in a steep wave, carried out by Longuet-Higgins in 1986

In a laboratory wave channel (width 60 cm, length 40 m, containing water of mean depth 35 cm), a small wooden bead, painted white, was floated on the water surface at a distance of about 20 m from the wavemaker. Viewed through a window in the side of the tank, a time exposure of the bead's path was taken.



QUESTIONS The following issues are of interest:

- Using formally quadratic quantities in the wave amplitude within the framework of linear theory, one can compute the average flow of energy and infer that the water particles in the fluid experience on average a net displacement in the direction in which the waves are propagating, the corresponding mean rate of movement being known as the Stokes drift. It therefore seems probable that most particles do not follow closed orbits. What artifacts are at the origin of the classical linear theory? How does one explain the first photograph?
- Is there some regular pattern of the trajectories or is the behavior rather chaotic and all we can hope for is to understand the motion after averaging? More precisely, the open loops obtained experimentally by Longuet-Higgins for steep waves are the rule or the rather the exception?

A SURVEY OF THE CLASSICAL LINEAR THEORY One obtains by dropping all nonlinear terms the linear wave solution

$$\begin{cases} \eta(x-ct) = \varepsilon\, d\, \cos(kx-\omega t), \\ u(x-ct,y) = \varepsilon\, \omega d\, \frac{\cosh(k[y+d])}{\sinh(kd)}\, \cos(kx-\omega t)\,, \\ v(x-ct,y) = \varepsilon\, \omega d\, \frac{\sinh(k[y+d])}{\sinh(kd)}\, \sin(kx-\omega t)\,, \\ P(x-ct,y) = P_{atm} - \rho g y + \varepsilon\, \rho g d\, \frac{\cosh(k[y+d])}{\cosh(kd)}\, \cos(kx-\omega t), \end{cases}$$

of wavelength $\lambda > 0$. Here

$$k = \frac{2\pi}{\lambda}, \qquad \omega = \sqrt{gk\,\tanh(kd)},$$

are the wavenumber, respectively the frequency, and the dispersion relation

$$c = \frac{\omega}{k} = \sqrt{g\,\frac{\tanh(kd)}{k}}$$

determines the speed c of the linear wave.

The motion of the particle $(x(t), y(t))$ beneath this linear wave is described by the nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = u = M \cosh(k[y+d]) \cos k(x-ct), \\ \dfrac{dy}{dt} = v = M \sinh(k[y+d]) \sin k(x-ct), \end{cases}$$

with initial data (x_0, y_0), where $M = \dfrac{\varepsilon \omega d}{\sinh(kd)}$. The classical approach pursued in the books [Crapper, Lamb, Milne-Thompson, Stoker] is to seek approximations of the solution in terms of the small parameter M. We restrict our attention to a the fixed time interval $[0, T]$, where $T = \lambda/c > 0$ is the wave period. Since y belongs *a priori* to a bounded set due to $-d \leq y \leq d\varepsilon$, we readily obtain that

$$x(t) - x_0 = O(M), \quad y(t) - y_0 = O(M), \qquad t \in [0, T],$$

where $O(M)$ denotes an expression of order M.

Using the mean-value theorem, we get

$$\begin{cases} \dfrac{dx}{dt} = M\, \cosh(k[y_0+d])\, \cos(kx_0 - \omega t) + O(M^2), \\ \dfrac{dy}{dt} = M\, \sinh(k[y_0+d])\, \sin(kx_0 - \omega t) + O(M^2), \end{cases}$$

since $\omega = kc$. Neglecting terms of second order in M, we find that

$$\begin{cases} \dfrac{dx}{dt} \approx M\, \cosh(k[y_0+d])\, \cos(kx_0 - \omega t), \\ \dfrac{dy}{dt} \approx M\, \sinh(k[y_0+d])\, \sin(kx_0 - \omega t)\,. \end{cases}$$

By integration we obtain

$$\begin{cases} x(t) \approx x_0 - \frac{M}{\omega}\, \cosh(k[y_0+d])\, \sin(kx_0 - \omega t), \\ y(t) \approx y_0 + \frac{M}{\omega}\, \sinh(k[y_0+d])\, \cos(kx_0 - \omega t)\Big). \end{cases}$$

Thus

$$\frac{[x(t)-x_0^*]^2}{\cosh^2(k[y_0+d])}+\frac{[y(t)-y_0^*]^2}{\sinh^2(k[y_0+d])}\approx\frac{M^2}{\omega^2},$$

which is the equation of an ellipse: to a first-order approximation the water particles move in closed elliptic orbits, the centre of the ellipse being (x_0^*, y_0^*), with

$$x_0^*=x_0+\frac{M}{\omega}\sin(kx_0),\quad y_0^*=y_0-\frac{M}{\omega}\cos(kx_0).$$

Criticism of the classical linear approach: Within linear water wave theory, the system of ordinary differential equations system describing the motion of the particles is inherently nonlinear, and in the classical approach a further linearization of this system is performed. One can hardly expect this 'brute-force' approach to yield an accurate description of the subtleties of the particle path motion, especially since the resulting outcome of closed orbits throughout the flow is a pattern that is easily destroyed by small perturbations.

Nonlinear theory — Stokes waves

DEFINITION: *Smooth periodic travelling wave solutions, with a single crest and trough per period and with a profile that is decreasing from crest to trough. Moreover, η, u, P are symmetric while v is antisymmetric about the crest, and we require that there is no underlying current.*

The mean current κ is the average horizontal current in the water,

$$\kappa = \frac{1}{L} \int_0^L u(x, y_0)\, dx$$

(the average of u on any horizontal line $y = y_0$ below the wave trough level). Physically we can imagine that swell originating from a distant storm enters a region of water in uniform flow. The case $\kappa = 0$ corresponds to swell entering a region of still water: Stokes' original definition of the wave speed (as the mean velocity in the moving frame of reference in which the wave is stationary).

There are no stagnation points in a smooth Stokes flow:

$$u < c$$

throughout the fluid. This is consistent with experimental data, indicating that in general the horizontal motion of individual water particles is slower than the propagation speed of the wave. On the other hand, Stokes' famous wave of greatest height, for which the symmetric free surface is not continuously differentiable, presents a stagnation point at the crest, where $u = c$, and the profile has a corner with an angle of 120° cf. [Amick & Toland, Plotnikov].

REMARK: The fact that $u = c$ and $v = 0$ at the crest in Stokes' wave of greatest height leads to the phrase "stagnation point at the crest". However, the particle at the crest does not actually move with the wave, being left behind as the wave propagates. We actually have an apparent stagnation point.

Stream function formulation

Due to mass conservation we can introduce in the moving frame

$$(x, y) \mapsto (x - ct, y)$$

the stream function

$$\psi(x, y) = m + \int_{-d}^{y} [u(x, s) - c]\, ds$$

with the following properties:

- ψ has period L in the x-variable and $\Delta\psi = 0$;
- $\psi_x = -v, \quad \psi_y = u - c$;
- ψ is constant on $y = \eta(x)$ and on $y = -d$. If m is the relative mass flux

  $$m = \int_{-d}^{\eta(x)} \Big(c - u(x, y)\Big)\, dy > 0,$$

  then $\psi = 0$ on $y = \eta(x)$.

Euler equation $\Longleftrightarrow$ Bernoulli's law: the expression

$$E = \frac{(c-u)^2 + v^2}{2} + gy + P$$

is constant thoughout the flow. We can reformulate the free boundary problem as

$$\begin{cases} \Delta\psi = 0 \quad \text{in} \quad -d < y < \eta(x), \\ |\nabla\psi|^2 + 2g(y+d) = Q \quad \text{on} \quad y = \eta(x), \\ \psi = 0 \quad \text{on} \quad y = \eta(x), \\ \psi = m \quad \text{on} \quad y = -d, \end{cases}$$

which is to be solved in the class of functions that are of period $L = 2\pi$ in the x-variable. Q and m are physical constants (head, relative mass flux).

Main difficulties: nonlinear character, surface profile η unknown.

Conformal change of variables

Introduce the velocity potential

$$\phi(x, y) = \int_0^x [u(l, -d) - c]\, dl + \int_{-d}^y v(x, s)\, ds,$$

with the following properties:

- $\phi_x = u - c$, $\phi_y = v$;
- $\Delta\phi = 0$;
- ϕ is odd in the x-variable and $\phi = 0$ on the crest line $x = 0$;
- $\phi(x, y) + cx$ has period 2π in x.

The conformal hodograph change of variables

$$q = -\phi(x, y), \quad p = -\psi(x, y),$$

transforms the free boundary problem into a nonlinear boundary problem for the harmonic function

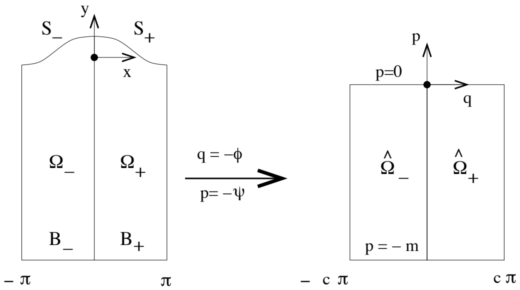
$$h(q, p) = y + d$$

in a fixed rectangular domain.

The transformed boundary problem is

$$\begin{cases} \Delta_{q,p}\, h = 0 \quad \text{for} \quad -(c-\kappa)\pi < q < (c-\kappa)\pi, \quad -m < p < 0, \\ h = 0 \quad \text{on} \quad p = -m, \\ 2\,(Q - gh)(h_q^2 + h_p^2) = 1 \quad \text{on} \quad p = 0, \end{cases}$$

for h even and periodic of period $2\pi\, c$ in the q-variable.



The conformal change of variables .

Particle paths beneath a Stokes wave

P is superharmonic:

$$\Delta P = -\psi_{xx}^2 - 2\psi_{xy}^2 - \psi_{yy}^2 \leq 0.$$

Since $P_y = -g < 0$ on $y = -d$ by Hopf's maximum principle the minimum of P is attained only along the free surface $y = \eta(x)$, where $P = P_{atm}$.

u and v harmonic in $(x, y) \Rightarrow$ harmonic in (q, p) due to the conformal change of variables. Thus u_q is harmonic. Futhermore, using Hopf's maximum principle and the boundary conditions, we get:

- $u_q = 0$ along the lateral sides of $\hat{\Omega}_+$;
- $u_q < 0$ on the lower boundary $\hat{B}_+$ of $\hat{\Omega}_+$;
- $u_q < 0$ on the top boundary $\hat{S}_+$.

The strong maximum principle yields $u_q < 0$ in $\hat{\Omega}_+$.

Thus, due to symmetry, we get: u is a strictly increasing function of x along any streamline in $\overline{\Omega}_-$ and a strictly decreasing function of x along any streamline in $\overline{\Omega}_+$.

In the physical variables, the particle path $\{(X(t),\ Y(t))\}_{t\geq 0}$ starting at (X_0, Y_0) solves

$$\begin{cases} X'(t) &= u(X-ct, Y), \\ Y'(t) &= v(X-ct, Y), \end{cases}$$

with initial data $(X(0),\ Y(0))=(X_0, Y_0)$. This corresponds to the streamline $x(t)=X(t)-ct,\ y(t)=Y(t)$ in the moving frame, and to the autonomous Hamiltonian system (with Hamiltonian function ψ)

$$\begin{cases} x'(t) &= u(x,y)-c, \\ y'(t) &= v(x,y)\,. \end{cases}$$

As $u-c<0$, for any particle path there is a time, say $t=0$, when $x=\pi$, and another time, say $t=\theta>0$ when $x=-\pi$. θ is the elapsed time (the time it takes to traverse one period in the moving plane).

The image of the streamline in the conformal frame is a streamline, given by $\psi(x, y) = -p$ or $y = y(x)$, and the elapsed time is

$$\theta(p) = \int_{-\pi}^{\pi} \frac{dx}{c - u(x, y(x))} > \frac{2\pi}{c}.$$

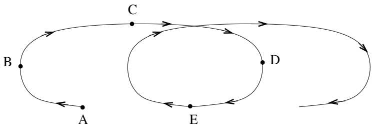
By the drift of a particle we mean the net horizontal distance moved by the particle between its positions below two consecutive troughs (or crests), that is,

$$X(\theta) - X(0) = c\,\theta - 2\pi = X(t+\theta) - X(t), \qquad t \in \mathbb{R}.$$

This corresponds to one period in the moving plane. The drift is positive if the particle always moves in the direction of wave propagation, and negative if it moves in the reverse direction. Zero drift characterizes a closed particle path that corresponds to a solution of period $\theta = \dfrac{2\pi}{c}$. We prove that the drift of a particle strictly decreases with depth, being positive at the flat bed.

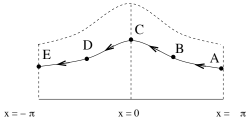
Particle path above the flat bed for a Stokes wave .

(i) In the physical frame at A and E the wave trough is right above the particle , while at C the wave crest is right above .

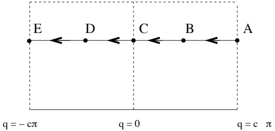


Physical frame .

(ii) Depiction of the corresponding motion in the moving frame and in the conformal frame: in both cases the motion is to the left . The free surface is also drawn.



Moving frame .



Conformal frame .

Conclusions

- The above results, due to Constantin (Invent. Math., 2006) and Constantin & Strauss (Comm. Pure Appl. Math., 2010), show that there are no closed particle paths.
- The pattern observed experimentally by Longuet-Higgins for the surface particle trajectories in a steep Stokes wave is the general pattern for any particle above the flat bed.
- The Stokes drift is not a phenomenon noticeable just on average: all particles are looping in the direction of wave propagation.

2. Wave breaking

"Truth in science can be defined as the working hypothesis best suited to open the way to the next better one."
Konrad Lorenz (Nobel Prize in Physiology/Medicine, 1973)

DEFINITION (WHITHAM) *For a two-dimensional surface wave,* wave breaking *occurs at time $T > 0$ if the wave remains bounded but its slope becomes unbounded.*

This specification encompasses both types of breaking waves:

- spilling breakers (the crests become peaky and start to develop whitecaps — a turbulent mass of air and water running down the front slope of the wave);
- plunging breakers (the leading slope along the crest line of the wave becomes vertical, the top leaps forward and a veritable cascade tumbles down in front of the wave).

Spilling breaker in front of a plunging breaker (1)

The first snapshot shows the verticality developing in the slope of the plunging breaker. The wind, from the beach towards the ocean, causes the spray to be blown 'backwards'.

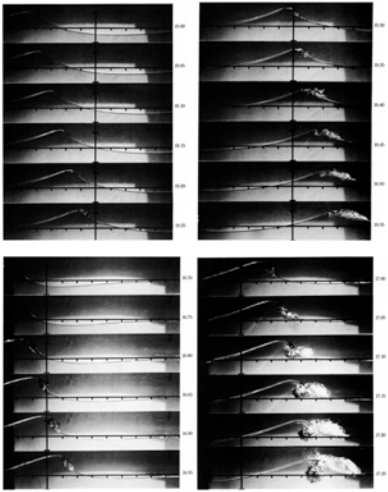
Spilling breaker in front of a plunging breaker (2)

In the second snapshot we clearly see the overturning, plunging breaker with an overhanging profile.

Spilling breaker in front of a plunging breaker (3)

In the third snapshot the plunging section of the wave has fallen down onto the surface that exists between this wave and the preceding one. In all three snapshots, the spilling breaker is essentially unchanged.

A single spilling/plunging wave in the laboratory [From R. J. Rapp and W. K. Melville, Phil. Trans. Roy. Soc. London A, 1990]



The snapshots on the top indicate the evolution of a spilling breaker, with a plunging breaker on the bottom.

AIM: Insight into the wave breaking phenomenon for unidirectional water waves propagating over a flat bed.

DIFFICULTY: The study of breaking waves, using the governing equations for water waves, has proved intractable.

PROGRESS: Perform approximations that lead to simplified model equations. For this, one has to write the governing equations in non-dimensional form so that the terms can be compared and one can give a meaning to "small with respect to".

FEATURES:

- within the confines of the linear theory, one cannot cope with the wave-breaking phenomenon;
- the weakly nonlinear theory of shallow water waves of small amplitude (e.g. the Korteweg-de Vries equation) does not accommodate breaking waves;
- nonlinearity must play a dominant role (Camassa-Holm regime of shallow water waves of moderate amplitude).

Non-dimensionalisation

The governing equations for irrotational two-dimensional water waves read, in non-dimensional form,

$$\begin{cases} \delta^2 \Phi_{xx} + \Phi_{yy} = 0 & \text{in} \quad \Omega(t), \\ \Phi_y = 0, & \text{on} \quad y = -1, \\ \zeta_t + \zeta_x \Phi_x - \dfrac{1}{\delta^2} \Phi_y = 0 & \text{on} \quad y = \varepsilon\,\zeta(x,t), \\ \Phi_t + \dfrac{\varepsilon}{2} \Phi_x^2 + \dfrac{\varepsilon}{2\delta^2} \Phi_y^2 + \zeta = 0 & \text{on} \quad y = \varepsilon\,\zeta(x,t)\,. \end{cases}$$

Here $x \mapsto \varepsilon\,\zeta(x,t)$ parametrizes the free surface at time t,

$$\Omega(t) = \{(x,y) :\, -1 < y < \varepsilon\,\zeta(x,t)\}$$

is the fluid domain delimited above by the free surface and below by the flat bed $y = -1$, and $\Phi(\cdot,\cdot,t) : \Omega(t) \to \mathbb{R}$ is the velocity potential associated with the flow, so that the two-dimensional velocity field is given by $(u,v) = (\Phi_x, \Phi_y)$.

The dimensionless amplitude parameter ε and the dimensionless shallowness parameter δ, are defined by

$$\varepsilon = \frac{a}{d}, \qquad \delta = \frac{d}{\lambda},$$

where λ, a are the typical wavelength and amplitude, respectively.

Generally:

- $\varepsilon \to 0$ means small amplitude;
- $\delta \to 0$ means shallow water.

CAUTION: Shallow water does not mean small depth – it is a measure of the relative sizes of wavelength and average depth. For example, the propagation of the 2004 Boxing Day tsunami across the Indian Ocean was characterized by

$$a \approx 1\,\text{m}, \ \lambda \approx 100\,\text{km}, \ d \approx 4\,\text{km},$$

with $\delta \approx 4 \cdot 10^{-2}$ and $\varepsilon \approx 25 \cdot 10^{-5}$.

In the shallow water regime $\delta \to 0$, making assumptions on the respective size of ε and δ leads to (simpler) shallow water models.

THE SHALLOW-WATER, SMALL-AMPLITUDE REGIME Noticing the way in which ε and δ appear in the governing equations, the regime

$$\delta \ll 1, \quad \varepsilon = O(\delta^2),$$

arises naturally. At first order, one obtains the linear wave equation

$$\zeta_{tt} - \zeta_{xx} = 0 \quad \text{on} \quad y = 0,$$

with the general solution

$$\zeta(x,t) = f_+(x-t) + f_-(x+t),$$

with the wave profile $f_\pm$ moving with unchanged shape to the right/left at constant unit speed, the corresponding dimensional speed being $\sqrt{gd}$.

In the shallow-water, small-amplitude regime, the right-going wave satisfies the Korteweg-de Vries (KdV) equation

$$\zeta_t + \zeta_x + \varepsilon \frac{3}{2}\, \zeta \zeta_x + \delta^2\, \frac{1}{6}\, \zeta_{xxx} = 0\,,$$

neglecting terms of order $O(\delta^4)$. The evolution in time of the horizontal velocity field $u = \zeta + O(\varepsilon)$ is also governed by KdV.

ADVANTAGES: rich integrable structure (infinite-dimensional Hamiltonian system), infinitely many integrals of motion, solitary waves are solitons (recover speed and shape after interaction), re-expression of geodesic flow.

DRAWBACK: integrals of motion control all Sobolev norms $\Rightarrow$ global existence (no breaking waves).

THE SHALLOW-WATER, MODERATE-AMPLITUDE REGIME For waves of moderate amplitude, characterized by larger values of ε, it is natural to investigate the following scaling:

$$\delta \ll 1, \qquad \varepsilon = O(\delta).$$

At the depth $1/\sqrt{2}$, the evolution of the scaled horizontal velocity U is governed by the Camassa-Holm equation

$$U_t + \kappa U_x + 3UU_x - U_{txx} = 2U_x U_{xx} + UU_{xxx},$$

while at the depth $\sqrt{23}/6$ we find the Degasperis-Procesi equation

$$U_t + \widehat{\kappa} U_x + 4UU_x - U_{txx} = 3U_x U_{xx} + UU_{xxx},$$

neglecting terms of order $O(\delta^4)$ cf. Johnson (J. Fluid Mech., 2002) and Constantin & Lannes (Arch. Rat. Mech. Anal., 2009).

ADVANTAGES: rich integrable structure (infinite-dimensional Hamiltonian system), infinitely many integrals of motion, solitons, re-expression of geodesic flows.

The evolution equation for the free surface waves In this regime, to order $O(\delta^4)$, the equation for the evolution of the free surface is

$$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\,\zeta\zeta_x - \frac{3}{8}\,\varepsilon^2\,\zeta^2\zeta_x + \frac{3}{16}\,\varepsilon^3\,\zeta^3\zeta_x + \frac{\delta^2}{12}\,\zeta_{xxx} - \frac{\delta^2}{12}\,\zeta_{xxt} + \frac{7\varepsilon\delta^2}{24}(\zeta\zeta_{xxx} + 2\,\zeta_x\zeta_{xx}) = 0\,.$$

Unlike KdV, this differs from the equations for the horizontal fluid velocity at some depth.

ADVANTAGES: The integral of motion

$$\int_{\mathbb{R}} \left(\zeta^2 + \frac{\delta^2}{12}\zeta_x^2\right) dx$$

ensures the boundedness of the surface wave profile. If the initial profile is steep enough, then the slope becomes unbounded in finite time cf. Constantin & Lannes (Arch. Rat. Mech. Anal., 2009). Therefore the equation models breaking waves.