Dissipative solutions of the Euler equations

Camillo De Lellis

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The Euler and Navier-Stokes equations

They describe the motion of an incompressible fluid under some assumptions.
The unknowns are the pressure (a scalar field) and the velocity.

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \nu \Delta v$$
$$\operatorname{div} v = 0$$

$\nu > 0$ Navier Stokes
$\nu = 0$ Euler

The i-th component of the advective term $\operatorname{div}(v \otimes v)$ is given by

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The Euler and Navier-Stokes equations II

In this talk we will consider solutions which are defined on the entire 3-dimensional (resp. 2-dimensional) space and over some time interval I.

In several occasions we will consider periodic solutions and thus the domain of definition will be $\mathbb{T}^3 \times I$ (or $\mathbb{T}^2 \times I$).

I might be

- a bounded interval or a half line; in this case the left endpoint will be 0 and the equations will be complemented with an initial condition (Cauchy problem):

$$v(0, \cdot) = v_0$$

- the entire real line (ancient solutions)

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The Euler equations were derived more than 250 years ago (by Euler!) The Navier-Stokes equations date back to the middle of the 19th century.

Nonetheless several fundamental and outstanding open questions are still open: the most famous one is the blow-up problem for 3-dimensional solutions of the Cauchy problem.

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The conservation of energy

If (v, p) is a C^1 solution, we can scalar multiply the first equation by v:

$$\partial_t \frac{|v|^2}{2} + \mathrm{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) = \nu \left(\Delta \frac{|v|^2}{2} - |Dv|^2\right)$$

(NB: C^1 obviously enough for $\nu = 0$ (Euler). Less obvious when $\nu > 0$: use for instance the regularity theory for Navier-Stokes.)

Integrate in space (and by parts!) to derive the dissipation law for the kinetic energy:

$$\frac{d}{dt} \int |v|^2(x,t)\,dx = -\nu \int |Dv|^2(x,t)\,dx \qquad (1)$$

$\nu = 0$: classical solutions of Euler preserve the total kinetic energy.

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Weak solutions of the Euler equations

Three possible definitions of generalized solutions:

1: use the theory of distributions to define derivatives. Assume square summability of v ($v \in L^2$) to safely define $v \otimes v$.

2: use Fourier series (periodic setting) in space and reduce the PDE to an (infinite-dimensional) system of ODEs for the Fourier coefficients. The minimal assumption to give a meaning: $v(t, \cdot) \in L^2$ (with some uniformity in t).

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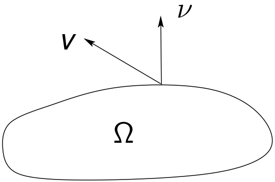
Balance of mass: the flux fluid leaving/entering Ω through $\partial\Omega$ is 0.



Conclusion: $$\int_{\partial\Omega} v \cdot \nu = 0$$

Balancing the momentum: $$\frac{d}{dt}\int_{\Omega} v = \int_{\partial\Omega} v(v\cdot\nu) + \int_{\partial\Omega} p\nu$$

These integral identities make sense if, for instance $(v, p) \in C$.

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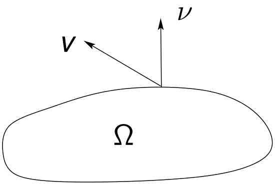

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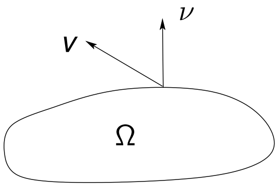

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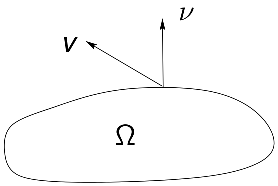

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The Scheffer-Shnirelman nonuniqueness

All these notions are equivalent and from now on: weak = generalized.

Theorem (Scheffer 1993)

There are compactly supported nonzero weak solutions in $L^2(\mathbb{R}^2 \times \mathbb{R})$.

A different proof in the periodic setting given by Shnirelman in 1998.

Obviously these solutions do not preserve the total kinetic energy.

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Theorem (D-Székelyhidi 2007)

There are compactly supported nontrivial bounded weak solutions in any space dimension.

D-S 2009: There are weak a solutions which “behave in all possible ways” in terms of local/global energy conservation. None of the criteria proposed so far in the literature restores uniqueness of weak solutions to the Cauchy problem if the initial data are discontinuous.

Székelyhidi 2011: this remains true even for fairly mild discontinuities.

Similar conclusions hold for other equations of fluid dynamics, where analogous methods can be used: Cordoba-Faraco-Gancedo, Shvidkoy, Wiedemann, Chiodaroli, ...

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Differential inclusions

Our 2007 paper plunged Scheffer's nonuniqueness Theorem in a long tradition of counterintuitive examples in differential inclusions and in differential geometry.

In the theory of differential inclusions you are looking at problems of the following type.

Problem

Given a set K of $k \times n$ matrices study maps $u : \mathbb{R}^n (\text{or } \Omega \subset \mathbb{R}^n) \to \mathbb{R}^k$ such that

$$\nabla u(x) \in K \qquad \text{for all } x \in \Omega. \tag{2}$$

It happens in several interesting situations that C^1 solutions are not so interesting because they are forced to be affine. In these cases we can look at Lipschitz solutions (which are differentiable a.e.!) and we turn (2) into

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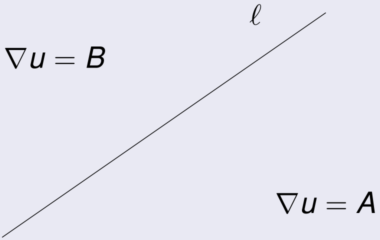
Differential inclusions II

Let us look at two "cousins" of the D-S Theorem.

Exercise

Consider two 2×2 matrices A and B: is there a Lipschitz planar map $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $\nabla u = A$ "on the left" and $\nabla u = B$ "on the right"?



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Solution: It exists if and only if the direction of ℓ is in the kernel of $A - B$.

However...

Theorem (Kirchheim 2003)

There are 2×2 matrices $A_1, A_2, \ldots, A_5$ and a Lipschitz map $u : \mathbb{R}^2 \to \mathbb{R}^2$ such that

- $\mathrm{rank}(A_i - A_j) = 2$;
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Another cousin: [Müller - Šverak 1999] there are solutions of nonlinear strongly elliptic systems of PDEs which are nowhere differentiable.

Connection between differential inclusions and elliptic systems??
I.e.: the Cauchy-Riemann equations are a differential inclusion!

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A toy example

$\Omega \subset \mathbb{R}^2$ smooth bounded open set. We look for planar (Lipschitz!) real-valued maps $\alpha : \Omega \to \mathbb{R}$ such that

$$|\nabla \alpha| = 1 \tag{4}$$

(+ maybe some boundary conditions...).

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Set up an iteration scheme producing $\varphi_0 \to \varphi_1 \to \varphi_2 \to \dots$

such that

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(where $\beta < 1$ is independent of k!). Prove convergence for φ_k.

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(where $\beta < 1$ is independent of k!). Prove convergence for φ_k.

A toy example

$\Omega \subset \mathbb{R}^2$ smooth bounded open set. We look for planar (Lipschitz!) real-valued maps $\alpha : \Omega \to \mathbb{R}$ such that

$$|\nabla \alpha| = 1 \tag{4}$$

(+ maybe some boundary conditions...).

PLAN: Start from some some smooth map φ_0 with $|\nabla \varphi_0| < 1$.

Set up an iteration scheme producing $\varphi_0 \to \varphi_1 \to \varphi_2 \to \dots$

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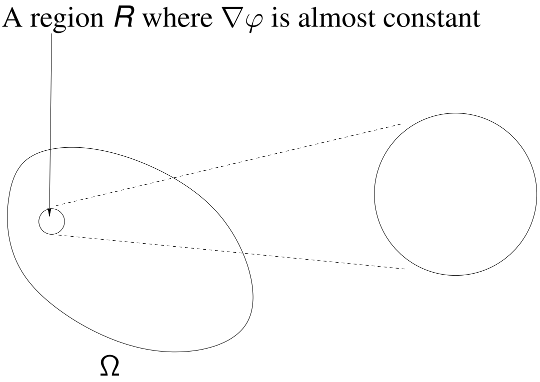
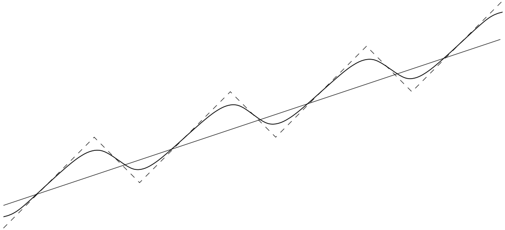
A toy example II

The iteration: from $\varphi_k = \varphi$ to $\varphi_{k+1} = \psi$



A toy example III

We make the slope of φ "steeper in average" in R by adding a periodic function which oscillates rapidly (in the direction of $\nabla\varphi$): we see below a cross section of φ and of the perturbed function $x \mapsto \varphi(x) + \frac{1}{\lambda}p(\lambda x)$



Next, cut off the perturbation to make it compactly supported in the region R:

$$\psi(x) = \varphi(x) + \frac{1}{\lambda}p(\lambda x)c(x)$$

(the cut-off c is compactly supported in R but mostly 1 in there).

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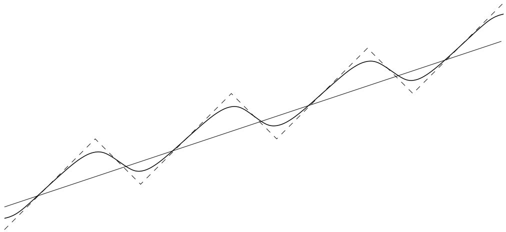

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We are now ready for the key computation:

$$\nabla\psi(x) = \nabla\varphi(x) + \underbrace{c(x)\nabla p(\lambda x)}_{\text{Improvement}} + \underbrace{\frac{1}{\lambda} p(\lambda x)\nabla c(x)}_{\text{Error}}$$

The Improvement "pushes" the slope towards 1 (at least in most places!).

The error can be made as small as we wish if λ is very large: this is not destroying what we gained with the Improvement.

Take care, do not get immediately to slope 1 (or above!) with the Improvement: for λ large we will keep the inequality $|\nabla\psi| < 1$.

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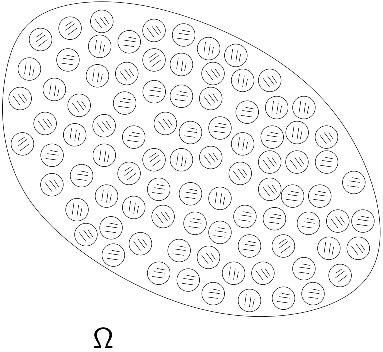
A toy example V

Repeat now this in many many small balls which cover a substantial portion of the region where $|\nabla\varphi|$ is "far" from 1.



From analysis to geometry

The upshot is: in all these “crazy” constructions the final (more or less counterintuitive) map is achieved through the addition of very fine oscillations to some underlying “subsolution”: the oscillations “pile up” and we reach the desired map only after infinitely many steps.

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The Nash-Kuiper Theorem

An older tradition of counterintitive construction indeed exists in differential geometry (Nash-Kuiper, Smale's paradox, Gromov, Eliashberg, ...).

Rather than trying to introduce the *h*-principle let me give an example, (maybe the "mother" of all these constructions?).
Consider a (smooth) Riemannian manifold (M, g): an isometry $u : M \to \mathbb{R}^N$ is a map preserving the length of curves.

In what follows we deal with C^1 maps which are also embeddings: isometric embeddings.

Corollary

Consider the standard sphere $(\mathbb{S}^2, \sigma)$ or the flat square $([0,1]^2, f)$. For any given $\varepsilon > 0$ there are C^1 isometric embeddings of these manifolds in a euclidean three-dimensional ball of radius ε, $B_\varepsilon(0) \subset \mathbb{R}^3$!

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Indeed the Theorem of Nash-Kuiper is much more general and much more precise: every short embedding (i.e. which shrinks the length of curves) of a compact Riemannian manifold can be uniformly approximated by C^1 isometric embeddings.

In the framework introduced by Gromov this can be translated into a "C^0-dense h-principle" (combining Nash-Kuiper with the Hirsch-Smale h-principle).

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Something like that holds for all the results mentioned in the theory of differential inclusions... With a big caveat:

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Let us go back to the Navier-Stokes equations and assume that the "viscosity" is small:

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There is a very large literature about the theory of turbulence both on the physical and mathematical side. But to my knowledge there is no rigorous proof of the existence of one single sequence of solutions v_{ε_k} to Navier-Stokes with

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Conjecture (Onsager 1949)

(A) Assume v is a (periodic) weak solution of the Euler equations satisfying an Hölder condition with exponent $\alpha > \frac{1}{3}$:

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Then the total kinetic energy of v is conserved:

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The solutions considered by Onsager are not differentiable... A physicist (with a Nobel prize in chemistry) considers indeed weak solutions in our modern sense in 1949.

And he gave a very rigorous definition, following “road 2” from some slides ago, i.e. via a Fourier series expansion.

Part (A) of the Conjecture has been proved by Eyink and Constantin-E-Titi in 1993. (see also Cheskidov-Constantin-Friedlander-Shvidkoy 2008)

Concerning Part (B), Scheffer’s example is the first rigorous instance (although the velocity is not even bounded!).

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Continuous dissipative solutions

Theorem (D-Székelyhidi 2011)

Let I be a compact interval and $e : I \to \mathbb{R}$ any given smooth positive function. Then there is a continuous solution (v, p) of the Euler equations in $\mathbb{T}^3 \times I$ such that

$$\int |v|^2(x,t)\,dx = e(t) \qquad \forall t \in I\,.$$

To our knowledge:

- This is the first example of its kind in the "analysis of PDEs" which lies in the "C^0 category" (which is instead customary in geometry);
- It relies upon "more complicated elliptic operators" (Δ and Δ^2) whereas all other "convex integration" constructions rely on "integrating" $\frac{d^k}{dr^k}$ (cp. with a question of Gromov).

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Hölder dissipative solutions

Theorem (D-Székelyhidi 2012)

Let I be a compact interval, $e : I \to \mathbb{R}$ any given smooth positive function and $\alpha < \frac{1}{10}$. Then there is a α-Hölder solution (v, p) of the Euler equations in $\mathbb{T}^3 \times I$ such that

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A funny coincidence?

[Borisov 1963, 2004] The Nash-Kuiper Theorem for isometric embeddings holds for $C^{1,\alpha}$ maps if α is sufficiently small (recall: $C^{1,\alpha}$ means C^1 + the differential of the map is α-Hölder). For instance

Theorem

There is an $\alpha_0 > 0$ with the following property. Consider the standard sphere $(\mathbb{S}^2, \sigma)$. For any given ε and any $\alpha < \alpha_0$ there are $C^{1,\alpha}$ isometric embeddings of these manifolds in a euclidean three-dimensional ball of radius ε, $B_\varepsilon(0) \subset \mathbb{R}^3$.

See [Conti-D-Székelyhidi 2009] for a shorter proof and more general results in this direction.

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