Irregular motion and global instability in Hamiltonian systems

6th European Congress of Mathematics
Kraków, July 2–7, 2012

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July 4th, 2012

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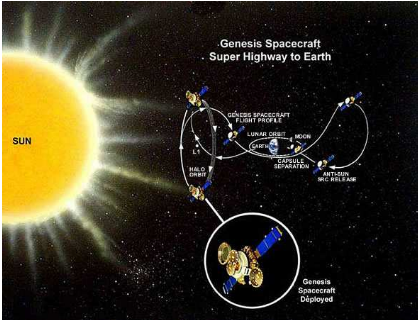
Outline

1. Set up
2. A priori unstable systems
3. An example of direct verification
4. A priori chaotic systems: geodesic flow
5. A priori chaotic systems: ERTBP
6. Proof for the example
    - I: A NHIM with transverse manifolds
    - II: Outer dynamics: Scattering map
    - III: Inner dynamics
    - IV: Construction of a transition chain



Diffusion: Global Instability

Main question:

Understand how small forces produce large effects in mechanical systems without friction

What is diffusion (or global instability)?

- Diffusion $\equiv$ Gaining lots of energy by applying small forces.
- Diffusion $\equiv$ Changes of order 1 in the actions (instabilities) for arbitrarily small perturbations of integrable systems.
- If a periodic perturbation is applied to a system; will the perturbation accumulate or will it average out?

Hamiltonian systems with more than 2 degrees of freedom

The main conjecture:
"Typical systems in action-angle variables have orbits whose actions change widely even if the systems are close to integrable"
Evidences:

- Mathematical:
  An example due to Arnold [Arnold64] (to be discussed later as an *a priori unstable system*)
- Numerical studies (Chirikov, Tennyson, Lieberman 75 on)
- Physical intuition (Fermi 34 on)

Stability or Instability?

Main Goals

Can we distinguish when perturbations accumulate and when they do not?

- Given a concrete system, can we say whether the perturbations accumulate or not?
- Can we design systems for which the perturbations accumulate (e.g satellites that use the gravitational energy to move...)
- Can we design systems for which the perturbations do not accumulate (particle accelerators, plasma devices...)

Poincaré's program

Poincaré's program to analyze dynamical systems

Given a concrete dynamical system:

1. Find landmarks that organize the long-term behavior (periodic orbits, invariant manifolds, homoclinic orbits, KAM tori, . . .
2. Perform a local analysis around them (normal forms, linearizations, . . .
3. Study how all this fits together (topology)

We obtain an skeleton of the dynamics. In particular, we may obtain regions of instability close to saddle invariant objects

Poincaré's program

In this talk, we will describe several combinations of invariant objects and their connections which

- Lead to large effects.
- Are persistent.
- Happen in near integrable systems.
- There are efficient algorithms to compute them.

The method of study that we will propose will require to identify *"roads"* in phase space in which the orbits move easily.
We will identify several combinations of objects which lead to diffusion. i.e. different mechanisms with different geometric intuition and different quantitative properties.

Tools

New and old Tools

Main tools we will use are standard tools accumulated over many years:

- Averaging methods
- KAM theory
- Persistence of normally hyperbolic invariant manifolds (NHIM)

And new ones:

- Two dynamics on the NHIM: inner map and scattering map
- Correctly aligned windows (with M. Gidea, R. de la LLave and P. Roldán)
- Computer assisted proofs (with M. Capiñski, P. Roldán, P. Zgliczyñski)

Warning: The effects considered happen only in ≥ 5 dimensions, so it will require some imagination in the presentation.

This talk

We are going to consider in this talk only two kinds of Hamiltonian systems:

- a priori unstable (2+1/2 or more degrees of freedom)
- a priori chaotic Hamiltonian systems with $2 + 1/2$ degrees of freedom ((Quasi)-periodic potentials in geodesic flows and Elliptic Restricted Three Body Problem (ERTBP).

Other systems under current research (Newer directions):

- Quantitative estimates for time diffusion in celestial mechanics, close to (saddle) Libration points (D-Gidea-Roldán)
- Computer assisted proofs of instability in celestial mechanics (Capiñski-D-Roldán-Capiñski-Zgliczyñski)
- Instability in non-conservative systems, like in: Computational neuroscience (D-Guillamon-Huguet), Reaction dynamics (Borondo-D-Roldán) ...

A priori unstable and a priori chaotic systems

- A priori unstable (Hamiltonian) systems:

$$H_\epsilon(p,q,I,\varphi,t) = H_0(p,q,I) + \epsilon h(p,q,I,\varphi,t;\epsilon)$$

For $\epsilon = 0$, $H_0(p,q,I)$ is autonomous ($H_0 = E$ =constant) integrable but with some saddle variables p, q. Typical example: one rotor (or more) plus one (or more) pendulum.

- A priori chaotic (Hamiltonian) systems:

$$H_\epsilon(p,q,t) = H_0(p,q) + \epsilon h(p,q,t;\epsilon)$$

For $\epsilon = 0$, $H_0(p,q)$ is autonomous non-integrable but with some saddle invariant object inside every level of energy $H_0 = E$ giving rise to chaotic motion inside $H_0 = E$. Typical example: geodesic flow on a manifold.

Main question: What happens to $E(t)$ for small $\epsilon \neq 0$? Is there global instability?: $E(t) - E(0) = \mathcal{O}(1)$ or even $E(t) \longrightarrow \infty$?

Instability for a priori unstable Hamiltonian systems

We consider a 2π-periodic in time perturbation of a pendulum and a rotor described by the non-autonomous Hamiltonian,

$$\begin{aligned} H_\epsilon(p,q,I,\varphi,t) &= H_0(p,q,I) + \epsilon h(p,q,I,\varphi,t;\epsilon) \\ &= P_\pm(p,q) + \tfrac{1}{2}I^2 + \epsilon h(p,q,I,\varphi,t;\epsilon) \end{aligned} \tag{1}$$

where $(p,q,I,\varphi,t) \in (\mathbb{R}\times\mathbb{T})^2 \times \mathbb{T}$ and

$$P_\pm(p,q) = \pm\left(\frac{1}{2}p^2 + V(q)\right) \tag{2}$$

and $V(q)$ is a 2π-periodic function. We will refer to $P_\pm(p,q)$ as the *pendulum*.

Note. This model [Chierchia-Gallavotti94] comes from a normal form around a single resonance of a nearly integrable Hamiltonian [D-Gutiérrez01] and originates in Poincaré and Arnold.

Main result for a priori unstable systems

Theorem (D-Llave-Seara06)

Consider the Hamiltonian (1) where V and h are uniformly $\mathcal{C}^{r+2}$ for $r \geq r_0$, sufficiently large. Assume also that

- **H1** *The potential $V : \mathbb{T} \to \mathbb{R}$ has a unique global maximum at $q = 0$ which is non-degenerate. Denote by $(q_0(t), p_0(t))$ an orbit of the pendulum $P_\pm(p, q)$ homoclinic to $(0, 0)$.*
- **H2** *The Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0)):*

$$\mathcal{L}(I, \varphi, s) = -\int_{-\infty}^{+\infty} (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) - h(0, 0, I, \varphi + I\sigma, s + \sigma; 0))d\sigma \quad (3)$$

  satisfies concrete non-degeneracy conditions.
- **H3** *The perturbation term h satisfies concrete non-degeneracy conditions.*

Then, there is $\epsilon^* > 0$ such that for $0 < |\epsilon| < \epsilon^*$, and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\widetilde{x}(t)$ of the system (1) such that for some $T > 0$,

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

Remark

Arbitrary excursions in the I variable can also be realized.

Hypotheses **H1**, **H2** and **H3** are $\mathcal{C}^2$ generic, so, the following short version of the Theorem also holds:

Theorem (D-Huguet09)

Consider the Hamiltonian (1) and assume that V and h are $\mathcal{C}^{r+2}$ functions which are $\mathcal{C}^2$ generic, with $r > r_0$, large enough. Then there is $\epsilon^* > 0$ such that for $0 < |\epsilon| < \epsilon^*$ and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\tilde{x}(t)$ of the system with Hamiltonian (1) such that for some $T > 0$

$$I(\tilde{x}(0)) \leq I_-^*; \qquad I(\tilde{x}(T)) \geq I_+^*.$$

Remark

A (non optimal) value of r_0 which follows from our argument is $r_0 = 242$.

Multidimensional a priori unstable Hamiltonian systems

Consider a periodic in time perturbation of n pendula and a d-dimensional rotor described by the non-autonomous Hamiltonian,

$$H(p, q, I, \varphi, t, \varepsilon) = P(p, q) + h(I) + \varepsilon Q(p, q, I, \varphi, t, \varepsilon), \qquad (4)$$

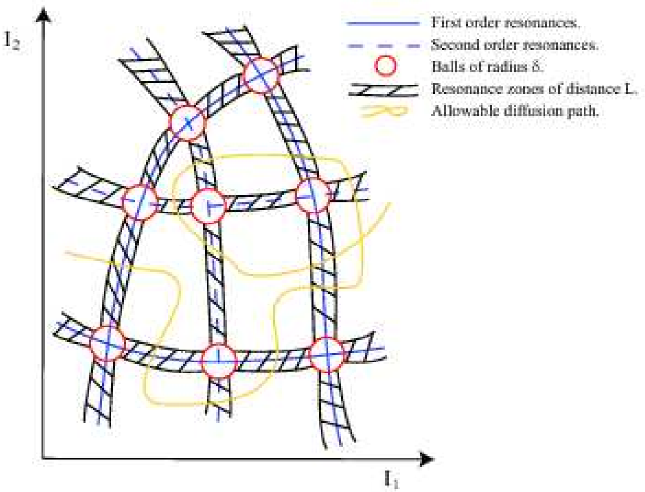
with $P(p, q) = \sum_{j=1}^{n} P_j(p_j, q_j)$, $P_j(p_j, q_j) = \pm\left(\frac{1}{2}p_j^2 + V_j(q_j)\right)$, where $I \in \mathcal{I} \subset \mathbb{R}^d, \varphi \in \mathbb{T}^d$, $\mathcal{I}$ an open set, $p, q \in \mathbb{R}^n$, $t \in \mathbb{T}^1$, and $P_j(p_j, q_j)$ is a *pendulum* for the saddle variables p_j, q_j. For $\epsilon = 0$, the d-dimensional action I remains constant. Under similar hypotheses as for $n = d = 1$,

Theorem (D-Llave-Seara12)

For every $\delta > 0$, there exists $\varepsilon_0 > 0$, such that for every $0 < |\varepsilon| < \varepsilon_0$, given $I_\pm \in \mathcal{I}$,there exists a solution $\tilde{x}(t)$ of (4) and $T > 0$, such that

$$|I(\tilde{x}(0)) - I_-| \leq C\delta \quad \textit{and} \quad |I(\tilde{x}(T)) - I_+| \leq C\delta \qquad (5)$$

- One can forget about δ and prescribe arbitrary paths on a set $\mathcal{I}^*$. This set $\mathcal{I}^*$ is described precisely in the course of the proof, and is determined by the non-degeneracy assumptions. The main idea is that $\mathcal{I}^*$ is obtained from the domain of definition, just eliminating some sets of codimension 2, like double resonances, from the open set where the intersection of stable and unstable manifolds of a normally hyperbolic invariant manifold is transversal.
- Codimension 2 objects do not separate the regions and can be contoured so that they do not obstruct the change along the paths.



I_2
I_1
First order resonances.
Second order resonances.
Balls of radius δ.
Resonance zones of distance L.
Allowable diffusion path.

Proof and other contributions

This problem of instability, also called Arnold diffusion, was posed first by Arnold in 1964, and there have been some other contributions, using geometrical or variational methods: [Chierchia-Gallavotti94-98], [Berti-Biasco-Bolle03], [Marco-Sauzin03], [Mather04], [Cheng-Yan04], [Gidea-Llave06], [Piftankin-Treschev07], [Kaloshin-Levi08].

Idea of the proof: use of two (or more) dynamics on $\widetilde{\Lambda}$

- Find a big invariant saddle object: a NHIM (normally hyperbolic invariant manifold: a global version of a center manifold) $\widetilde{\Lambda}$ with transverse associated stable and unstable manifolds along some homoclinic manifold Γ: $\mathcal{W}^u(\widetilde{\Lambda}) \pitchfork_\Gamma \mathcal{W}^s(\widetilde{\Lambda})$.
- Compute the invariant objects (typically tori $\mathcal{T}$) which may prevent instability for the inner dynamics of the NHIM.
- Compute the scattering map $S = S^\Gamma : \widetilde{\Lambda} \to \widetilde{\Lambda}$ on the NHIM associated to Γ and consider it as an outer dynamics on the NHIM (a second dynamics on Γ).
- Check that $S(\mathcal{T}_{I_i}) \pitchfork \mathcal{T}_{I_{i+1}}$ for a sequence of tori $\{\mathcal{T}_{I_i}\}_{i=1}^N$ with $|I_N - I_1| = \mathcal{O}(1)$, and construct a transition chain of whiskered tori, i.e. $\mathcal{W}^u(\mathcal{T}_{I_i}) \pitchfork \mathcal{W}^s(\mathcal{T}_{I_{i+1}})$.
- Standard shadowing methods provide an orbit that follows closely the transition chain.

An example of direct verification

Consider the Hamiltonian

$$H_\epsilon(p,q,I,\varphi,t) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \epsilon f(q) g(\varphi,t). \tag{6}$$

with

$$f(q) = \cos q, \tag{7}$$

and

$$g(\varphi,t) = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l}\cos(k\varphi - lt - \sigma_{k,l}). \tag{8}$$

with

$$\hat{\alpha}\rho^{(1+\beta)k}\ r^{(1+\beta)l} \leq |a_{k,l}| \leq \alpha\rho^k r^l, \tag{9}$$

where $0 < \rho, r < 1$ are real numbers to be chosen small (independently of ϵ^* and the interval $[I_-^*, I_+^*]$ of diffusion), and $0 \leq \beta < 1$. For instance,

$$g(\varphi,t) = \Re\left(\frac{1}{(1-\rho e^{i\varphi})(1-re^{-it})}\right).$$

Result for the example

Theorem (D-Huguet11)

Consider a Hamiltonian of the form (6) *where* $f(q)$ *is given by* (7) *and* $g(\varphi, t)$ *is any analytic function of the form* (7) *with non-vanishing Fourier coefficients satisfying* (9). *Assume also that either* $1.6\,|a_{1,0}/a_{0,1}| < 1$ *or* $1.6\,|a_{0,1}/a_{1,0}| < 1$.
Then, for any $I_+^* > 0$ *there exists* $\epsilon^* = \epsilon^*(I_+^*) > 0$ *such that for any* $0 < I_- < I_+ < I_+^*$ *and any* $0 < |\epsilon| < \epsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *of the Hamiltonian* (1) *such that for some* $T > 0$

$$I(0) \leq I_-; \qquad I(T) \geq I_+.$$

Remark on the perturbation

- If $g(\varphi, t) = G(t)$, the action I is constant.
- If $g(\varphi, t) = G(\varphi)$, Hamiltonian (6) is autonomous, i.e., H_ϵ is constant, so that only deviations of size $\sqrt{\epsilon}$ are possible for I.
- The same happens when $g(\varphi, t) = G(\psi)$, where $\psi = k_0\varphi + l_0 t$, introducing ψ as a new angular variable.

In these three cases, an infinite number of Fourier coefficients $a_{k,l}$ of the function $g(\varphi, t)$ in (7) vanish. This is one of the reasons why we have assumed conditions (9) for the harmonics $a_{k,l}$.
More general, and of course more technical, set of conditions for more general perturbations can be given explicitly.

(Quasi)-periodic perturbations of geodesic flows

Theorem ([D-Llave-Seara06])

Let M be a n-dimensional manifold, g a $\mathcal{C}^r$ metric on it (r sufficiently large). Assume:

H1 *There exists a closed geodesic "Λ" such that its corresponding periodic orbit $\hat{\Lambda}$ under the geodesic flow is hyperbolic.*

H2 *There exists another geodesic "γ" such that $\hat{\gamma}$ is a transversal homoclinic orbit to $\hat{\Lambda}$.*
That is, $\hat{\gamma}$ is contained in the intersection of the stable and unstable manifolds of $\hat{\Lambda}$, $W^s_{\hat{\Lambda}}$, $W^u_{\hat{\Lambda}}$, in the unit tangent bundle.
Moreover, we assume that the intersection of the stable and unstable manifolds of $\hat{\Lambda}$ is transversal along $\hat{\gamma}$. That is,

$$T_{\gamma(t)} W^s_{\hat{\Lambda}} + T_{\gamma(t)} W^u_{\hat{\Lambda}} = T_{\gamma(t)} \mathbf{S}_1 M, \quad t \in \mathbb{R}.$$

Abundance of Hypoteses **H1**, **H2**

Hipotheses **H1**, **H2** are abundant:

- They are generic on $\mathbb{T}^2$ [Morse24], [Hedlund32], [Mather94].
- They hold on any closed surface of genus bigger or equal than 2, if $r \geq 2 + \delta$, $\delta > 0$. [Katok82]).
- They are generic in the $\mathcal{C}^2$ topology for any closed surface [Contreras-Paternain02].

(Quasi)-periodic perturbations of geodesic flows

Let $\nu \in \mathbb{R}^d$ be Diophantine, $r \in \mathbb{N}$ be sufficiently large (depending on τ, the Diophantine exponent of ν).
Let g be a $\mathcal{C}^r$ metric on a compact manifold M, verifying hypotheses **H1**, **H2**, and $U : M \times \mathbb{T}^d \to \mathbb{R}$ a generic $\mathcal{C}^r$ function.
Consider the time dependent Lagrangian

$$L(q, \dot{q}, \nu t) = \frac{1}{2} g^q(\dot{q}, \dot{q}) - U(q, \nu t), \tag{10}$$

where g^q denotes the metric in $\mathbf{T}_q M$.
Then, the Euler-Lagrange equation of L has a solution $q(t)$ whose energy

$$E(t) = \frac{1}{2} g^q(\dot{q}(t), \dot{q}(t)) + U(q(t), \nu t),$$

tends to infinity as $t \to \infty$.

(Planar) elliptic restricted three body problem (ERTBP)

- Consider the motion of a particle q with zero mass (comet) under the attraction of two particles q_1 (Sun, with mass $1-\mu$) and q_2 (Jupiter, with mass μ), called *primaries*, which move in elliptic orbits with eccentricity e_0 around their center of mass.
- The motion of q is described by a time-periodic Hamiltonian system, with 2 and 1/2 degrees of freedom, with Hamiltonian

$$H(q,p,t;e_0,\mu) = \frac{p^2}{2} - \frac{(1-\mu)}{|q-q_1(t,e_0)|} - \frac{\mu}{|q-q_2(t,e_0)|}.$$

- We consider the motion of the particle q (comet) when it moves outside of the orbit of the primaries along nearly parabolic orbits.
- Parameters: $0<\mu<1$, $e_0 \geq 0$, small.

The two body problem: Sun-comet for $\mu = 0$

- When $\mu = 0$, the *Sun* is fixed at the origin: $q_1(t, e_0) = 0$
- The Sun q_1 and the comet q form the two-body problem.
- In polar coordinates: $q = (r\cos\alpha, r\sin\alpha)$, $\alpha \in \mathbb{T}$, $r \geq 0$, the Hamiltonian of the two body problem becomes

$$H_0(r, P_r, \alpha, G) = \frac{P_r^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r},$$

- H_0 is the energy and $G = P_\alpha$ is the angular momentum.
- H_0 and G are both first integrals of motion.
- If $H_0 = h < 0$, motions are elliptic with semi-major axis $a = 1/(-2h)$ and eccentricity $e = \sqrt{1 + 2hG^2}$.
- If $h = 0$ (which corresponds to $e = 1$) the motion is parabolic.
- The two-body problem is integrable.

Diffusion of the angular momentum G

In the elliptic restricted three body (ERTBP) problem we want to see that the angular momentum of the comet $G(t)$ can have *large changes* when the eccentricity $e_0 > 0$ and $\mu > 0$ are small enough:

Theorem (D-Kaloshin-Rosa-Seara12)

Given any $G_1, G_2 \gg 1$, there exist trajectories of the ERTBP whose angular momentum satisfies, for some $T > 0$:

$$G(0) < G_1 \qquad G(T) > G_2$$

Proven for $0 < \mu \ll e_0 \ll 1$ and any $1 \ll G_1, G_2 \ll 1/e_0$.
Likely (need still some work) for any $0 < e_0 < 1$ and $0 < \mu \ll 1$.

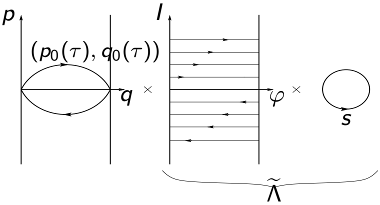
Remark

Two different scattering maps are used in the construction of the diffusing trajectories.

Sketch of the proof for the example

- **Part I**: Existence of a normally hyperbolic invariant manifold with associated stable and unstable manifolds.
- **Part II**: Outer dynamics.
- **Part III**: Inner dynamics.
- **Part IV**: Construction of a transition chain.

$\epsilon = 0$



- Normally hyperbolic invariant manifold (3D)

$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\}$$

- Invariant manifolds (4D):

$$W^s\widetilde{\Lambda} = W^u\widetilde{\Lambda} = \{(p_0(\tau), q_0(\tau), I, \varphi, s) : \tau \in \mathbb{R}, I \in [I_-, I_+], (\varphi, s) \in \mathbb{T}^2\}$$

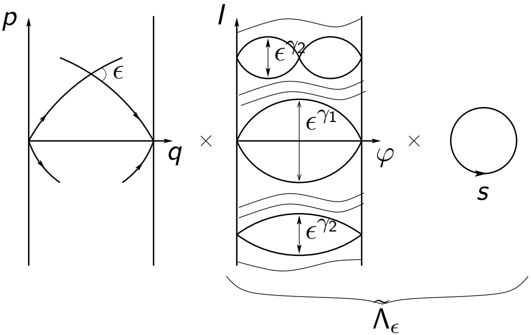
where

$$q_0(t) = 4 \arctan e^{\pm t}, \quad p_0(t) = 2/\cosh t.$$

is the separatrix for positive p of the standard pendulum

$$P(p, q) = p^2/2 + \cos q - 1.$$

$0 < \epsilon \ll 1$



- By the theory of NHIM, $\widetilde{\Lambda}$ persists to $\widetilde{\Lambda}_\epsilon$.
- $W^s\widetilde{\Lambda}_\epsilon$ and $W^u\widetilde{\Lambda}_\epsilon$ are ϵ-close to the unperturbed ones.
- $\Gamma_\epsilon \subset W^s\widetilde{\Lambda}_\epsilon \cap W^u\widetilde{\Lambda}_\epsilon$ homoclinic manifold.
- Using hypothesis **H2'**, $W^s\widetilde{\Lambda}_\epsilon \pitchfork W^u\widetilde{\Lambda}_\epsilon$ along Γ_ϵ.

Let us look at hypothesis **H2′** for the example:

H2' Given real numbers $I_- < I_+$, assume that for any value of $I \in (I_-, I_+)$ the map

$$\tau \in \mathbb{R} \mapsto \mathcal{L}(I, \varphi - I\tau, s - \tau)$$

has a non-degenerate critical point τ which is locally given by the implicit function theorem in the form

$$\tau = \tau^*(I, \varphi, s),$$

with τ^* a smooth function.

Then [D-Llave-Seara06] for ϵ small enough, there exists a locally unique point $\tilde{z}$ of the form

$$\tilde{z}(I, \varphi, s; \epsilon) = (p_0(\tau) + \mathcal{O}(\epsilon), q_0(\tau) + \mathcal{O}(\epsilon), I, \varphi, s),$$

such that $W^s(\tilde{\Lambda}_\epsilon) \pitchfork W^u(\tilde{\Lambda}_\epsilon)$ at $\tilde{z}$.

For the perturbation $\cos q\, g(\varphi, s)$, where

$$g(\varphi, s) = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l} \cos(k\varphi - ls - \sigma_{k,l}),$$

with $\hat{\alpha}\rho^{(1+\beta)k}\; r^{(1+\beta)l} \leq |a_{k,l}| \leq \alpha\rho^k r^l$, the Melnikov potential

$$\mathcal{L}(I, \varphi, s) = \frac{1}{2}\int_{-\infty}^{\infty} p_0^2(\sigma) g(\varphi + I\sigma, s + \sigma) d\sigma,$$

is given by

$$\mathcal{L}(I, \varphi, s) = \sum_{(k,l)\in\mathbb{N}^2} A_{k,l}(I) \cos(k\varphi - ls - \sigma_{k,l}),$$

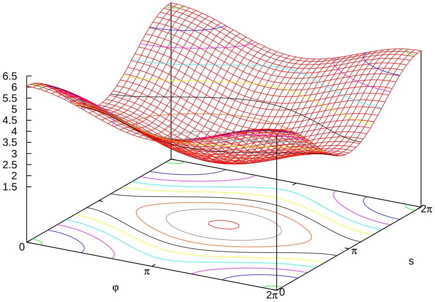
with

$$A_{k,l}(I) = 2\pi \frac{(kI - l)}{\sinh \frac{\pi}{2}(kI - l)} a_{k,l},$$

Graph and level curves of the Melnikov potential

$$\mathcal{L}(I,\varphi,s) = A_{0,0} + A_{1,0}\cos\varphi + A_{0,1}(I)\cos s + \mathcal{O}_2(\rho,r),$$

for $0 < A_{1,0} < A_{0,1} < 1$, where we have fixed $\sigma_{1,0} = \sigma_{0,1} = 0$



Four non-degenerate critical points: maximum $(0,0)$, minimum (π,π) and two saddles $(0,\pi)$, $(\pi,0)$.

Scattering map (outer map)

Ingredients:

- Consider the foliations $\mathcal{F}_{s,u}$:

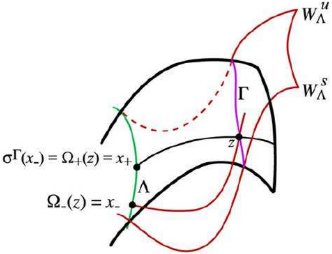
$$W^{s,u}_{\widetilde{\Lambda}_\epsilon} = \cup_{x\in\widetilde{\Lambda}_\epsilon} W^{s,u}_x$$

- Define the wave operators Ω_+, Ω_-:

$$\begin{array}{rccc} \Omega_\pm : & W^{s,u}_{\widetilde{\Lambda}_\epsilon} & \to & \widetilde{\Lambda}_\epsilon \\ & x & \mapsto & \Omega_\pm(x) \end{array}$$

  defined by $x \in W^{s,u}_{\Omega_\pm(x)}$.

- Ω_- is a diffeomorphism from Γ_ϵ to $H^{\Gamma_\epsilon}_- \equiv \Omega_-(\Gamma_\epsilon)$.



Define

$$S^\Gamma_\epsilon = \Omega^+ \circ (\Omega^{\Gamma_\epsilon}_-)^{-1}$$

- Scattering map (outer map):

$$\begin{array}{cccc} S_\epsilon : & H_-^{\Gamma_\epsilon} \subset \widetilde{\Lambda}_\epsilon & \to & H_+^{\Gamma_\epsilon} \subset \widetilde{\Lambda}_\epsilon \\ & x_- & \mapsto & x_+ \end{array}$$

  defined by $x_+ = S_\epsilon(x_-) \Leftrightarrow \exists\, z \in \Gamma_\epsilon$, such that

$$\mathrm{dist}(\Phi_t(z), \Phi_t(x_\pm)) \to 0 \qquad \text{for} \quad t \to \pm\infty$$

- S_ϵ is exact symplectic [D-Llave-Seara08]. Some examples in celestial mechanics numerically computed [Canalias-D-Masdemont-Roldán06], [D-Masdemont-Roldán08], [D-Gidea-Roldán12].

- Perturbative formula for the Hamiltonian $\mathcal{S}_\epsilon$ generating the deformation of the scattering map S_ϵ [D-Llave-Seara08]:

$$\mathcal{S}_\epsilon(I,\varphi,s) = -\mathcal{L}^*(I,\varphi - Is) + \mathcal{O}(\epsilon). \tag{11}$$

where the reduced Poincaré function $\mathcal{L}^*(I,\widetilde{\theta})$ is defined by

$$\mathcal{L}(I,\varphi - I\tau^*(I,\varphi,s), s-\tau^*(I,\varphi,s)) := \mathcal{L}^*(I,\varphi - Is). \tag{12}$$

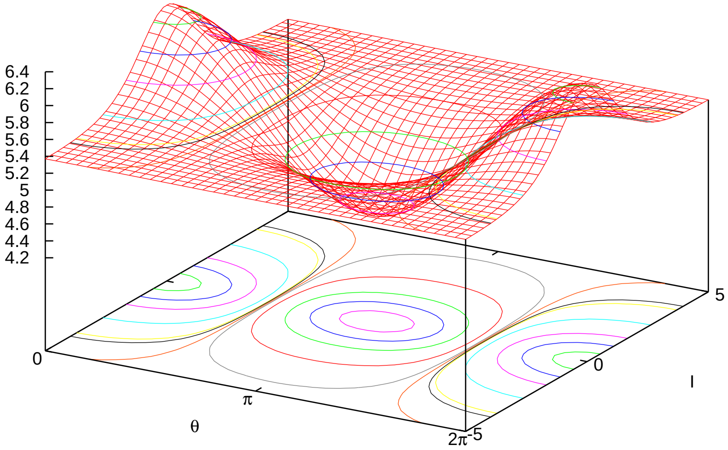
- The computation of S_ϵ up to first order gives

$$S_\epsilon(I,\varphi,s) = (I,\varphi,s) + \epsilon \mathrm{J}\nabla\mathcal{L}^*(I,\varphi - Is) + \mathcal{O}(\epsilon^2), \tag{13}$$

- The scattering map can jump distances of $\mathcal{O}(\epsilon)$ in terms of the variable I along the level curves of $\mathcal{L}^*(I,\widetilde{\theta})$.

Going back to our example. . .

Graph and level curves of the reduced Poincaré function $\mathcal{L}^*(I, \widetilde{\theta})$, where $\widetilde{\theta} = \varphi - Is$, for $a_{1,0} = 1/4$ and $a_{0,1} = 1/2$:



Main result for the inner dynamics

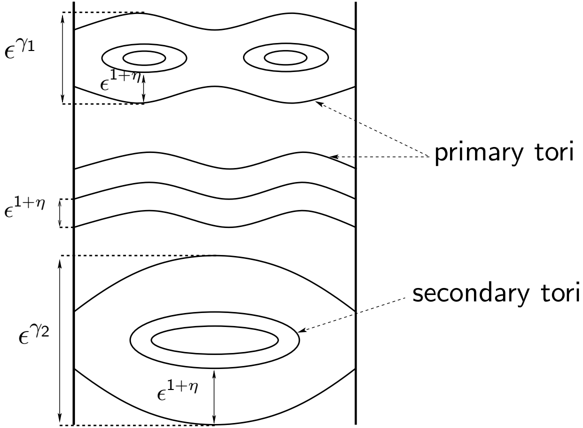
Theorem

Assume that $r > 2(m+1)^2$ and $m \geq 10$, then there exists a discrete sequence of invariant tori $\{\mathcal{T}_i\}_{i=1}^N$ in $\widetilde{\Lambda}_\epsilon$ such that:

- *They are distributed along the actions in the interval (I_-, I_+).*
- *They are $\mathcal{O}(\epsilon^{1+\eta})$-closely spaced in terms of the action variables, where $0 < \eta \ll 1$.*
- *They are given by the level sets defined by equation $F(I, \varphi, s; \epsilon) = E$, where F is a $\mathcal{C}^2$ function F which has different expressions depending on the region of the phase space where invariant tori lie:*
  - *Flat tori region. Primary KAM tori.*
  - *Big gaps region. Primary KAM tori and Secondary KAM tori.*

Proof: Averaging procedure + KAM Theorem.

Invariant objects in the NHIM $\widetilde{\Lambda}_\epsilon$



- We combine now the inner and the outer dynamics to construct a transition chain along $\widetilde{\Lambda}_\epsilon$:
  A sequence of whiskered tori $\{\mathcal{T}_i\}_{i=1}^N$ such that

$$W^u_{\mathcal{T}_i} \pitchfork W^s_{\mathcal{T}_{i+1}}$$

  Standard shadowing methods [Fontich-Martin00] provide orbits connecting arbitrary small neighborhoods of $\mathcal{T}_1$ and $\mathcal{T}_N$.

- We will use that

$$S_\epsilon(\mathcal{T}_i) \pitchfork_{\widetilde{\Lambda}_\epsilon} \mathcal{T}_{i+1} \Rightarrow W^u_{\mathcal{T}_i} \pitchfork W^s_{\mathcal{T}_{i+1}}$$

Invariant tori (primary and secondary) in the resonant region around $I = 0$ (red curves) given implicitly by the level sets of the function $F^*(I, \widetilde{\theta})$ with $k_0 = 1$, $l_0 = 0$ and $a_{1,0} = 1/2$. Images of these invariant tori (red curves) under the scattering map generated by the reduced Poincaré function $\mathcal{L}^*(I, \widetilde{\theta})$:

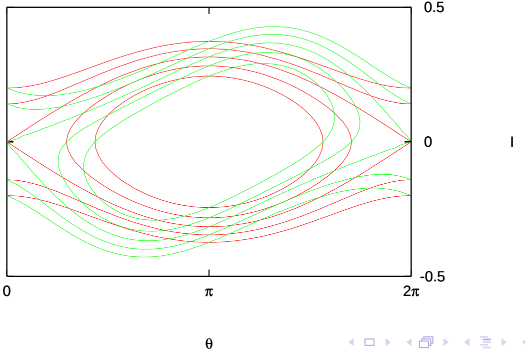


Illustration of how to combine the two dynamics to cross the big gaps region. Invariant tori for the inner dynamics (red curves) and invariant sets for the outer dynamics (blue curves). Inner dynamics is represented by dashed lines whereas outer dynamics is represented by solid lines.