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# Non-smooth Spline-Wavelet Expansions [2]

The simple methods for construction of chain of embedded spaces for the first order splines are done with local enlargement of irregular grid. Wavelet expansions are presented and independence of expansions with respect to the order of grid enlargement is established.

Let $X$ be irregular grid on the interval $(\alpha, \beta) \subset \mathbb{R}^1$, $X \stackrel{\text{def}}{=} \{x_j\}_{j \in \mathbb{Z}}$,

$$X: \quad \ldots < x_{-1} < x_0 < x_1 < \ldots,$$

where

$$\alpha = \lim_{j \to -\infty} x_j, \ \beta = \lim_{j \to +\infty} x_j, \qquad \forall j \in \mathbb{Z}.$$

We discuss the sets $S_j \stackrel{\text{def}}{=} (x_j, x_{j+1}) \cup (x_{j+1}, x_{j+2})$, $G = \bigcup_{j \in \mathbb{Z}} (x_j, x_{j+1})$.

Let $A \stackrel{\text{def}}{=} \{\mathbf{a}_j\}_{j \in \mathbb{Z}}$ be a system of column vectors $\mathbf{a}_j \in \mathbb{R}^2$, for which matrices $A_j \stackrel{\text{def}}{=} (\mathbf{a}_{j-1}, \mathbf{a}_j)$ are nonsingular for all $j \in \mathbb{Z}$; the mentioned system is called *complete chain* of vectors. Set of all complete chains is denoted by $\mathcal{A}$.

Let $\varphi(t)$ be two-component vector-function with linear independent components on arbitrary interval $(a, b) \subset G$. We define functions $\omega_j(t)$, $t \in G$, with approximation relations

$$\mathbf{a}_{i-1} \omega_{i-1}(t) + \mathbf{a}_i \omega_i(t) \equiv \varphi(t) \quad \forall t \in (x_i, x_{i+1}) \quad \forall i \in \mathbb{Z},$$

$$\omega_j(t) \equiv 0 \quad \forall t \in G \backslash S_j \quad \forall j \in \mathbb{Z}.$$

Let $\mathbb{S}$ be linear space: $\mathbb{S} = \mathbb{S}(X, A, \varphi) \stackrel{\text{def}}{=} Cl_p \ \mathcal{L}\{\omega_j\}_{j \in \mathbb{Z}}$; here $\mathcal{L}\{\ldots\}$ is linear hull of set that contained in braces and $Cl_p$ indicates closure in point convergence topology.

Suppose $\varphi \in C^s(\alpha, \beta)$ for a nonnegative integer $s$. By definition, put $\varphi_i^{(s)} \stackrel{\text{def}}{=} \varphi^{(s)}(x_i)$ $\forall i \in \mathbb{Z}$.

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**Theorem 1.** *If* $\{\varphi_j^{(s)}\}_{j\in\mathbb{Z}}$ *is a complete chain and* $\mathbf{a}_j \stackrel{\text{def}}{=} \varphi_{j+1}^{(s)}$*, then relations* $\omega_j^{(s)} \in C(\alpha,\beta)$ *are correct.*

For fixed $k\in\mathbb{Z}$ we discuss the knots $\widetilde{x}_j \stackrel{\text{def}}{=} x_j$ for $j\le k,$ and $\widetilde{x}_j \stackrel{\text{def}}{=} x_{j+1}$ for $j\ge k+1$, and consider a new grid $X_{\{k+1\}}: \quad \ldots < \widetilde{x}_{-1} < \widetilde{x}_0 < \widetilde{x}_1 < \ldots.$

By definition, put $\widetilde{G} \stackrel{\text{def}}{=} \bigcup_{j\in\mathbb{Z}}(\widetilde{x}_j,\widetilde{x}_{j+1})$, $\widetilde{S}_j \stackrel{\text{def}}{=} (\widetilde{x}_j,\widetilde{x}_{j+1})\cup(\widetilde{x}_{j+1},\widetilde{x}_{j+2})$. Let $\widetilde{A} \stackrel{\text{def}}{=} \{\widetilde{\mathbf{a}}_j\}_{j\in\mathbb{Z}}$ be complete chain of two-dimensional vectors. Suppose vector function $\varphi(t)$ is defined on the set $\widetilde{G}$. As before, taking into account the approximation relations

$$\sum_{j'\in\mathbb{Z}} \widetilde{\mathbf{a}}_{j'}\widetilde{\omega}_{j'}(t) = \varphi(t), \qquad \widetilde{\omega}_j(t)\equiv 0 \qquad \forall t\in \widetilde{G}\backslash\widetilde{S}_j,$$

we define functions $\widetilde{\omega}_j(t)$. The linear hull $\mathbb{S}(X_{\{k+1\}},\widetilde{A},\varphi)$ of functions $\{\widetilde{\omega}_j\}_{j\in\mathbb{Z}}$ is a space of splines of first order: $\widetilde{\mathbb{S}}_{k+1} = \mathbb{S}(X_{\{k+1\}},\widetilde{A},\varphi) \stackrel{\text{def}}{=} Cl_p\,\mathcal{L}\{\widetilde{\omega}_j\}_{j\in\mathbb{Z}}$.

Suppose that

$$\mathbf{a}_j = \widetilde{c}_j\widetilde{\mathbf{a}}_j \qquad \text{for} \quad j\le k-1, \qquad \mathbf{a}_{j+1} = \widetilde{c}_{j+1}\widetilde{\mathbf{a}}_j \qquad \text{for} \quad j\ge k, \tag{1}$$

where $\widetilde{c}_j$ are nonzero constants ($j\in\mathbb{Z}$).

Now we fix a chain $\widetilde{A}\in\mathcal{A}$ and consider the chain $A$ depending on the vector $\mathbf{a}_k=\mathbf{a}$, so that $A=A(\mathbf{a})$. Suppose $A(\mathbf{a})\in\mathcal{A}$. We put $\mathbb{S}(\mathbf{a}) \stackrel{\text{def}}{=} \mathbb{S}(X,A(\mathbf{a}),\varphi)$.

**Theorem 2.** *If relations (1) are fulfilled and* $\widetilde{A}$, $A(\mathbf{a})\in\mathcal{A}$*, then* $\widetilde{\mathbb{S}}_{k+1}\subset\mathbb{S}(\mathbf{a})$.

By definition, put $\omega(t) \stackrel{\text{def}}{=} (\ldots,\omega_{-2}(t),\omega_{-1}(t),\omega_0(t),\omega_1(t),\omega_2(t),\ldots)^T$, $\widetilde{\omega}(t) \stackrel{\text{def}}{=} (\ldots,\widetilde{\omega}_{-2}(t),\widetilde{\omega}_{-1}(t),\widetilde{\omega}_0(t),\widetilde{\omega}_1(t),\widetilde{\omega}_2(t),\ldots)^T$.

**Theorem 3.** *If the condition* $(B)$ *is valid, then* $\widetilde{\omega}(t) = \mathfrak{P}_{(k+1)}\omega(t)$*; here* $\mathfrak{P}_{(k+1)} \stackrel{\text{def}}{=} (\mathfrak{p}_{i,j})_{i,j\in\mathbb{Z}}$ *is infinite matrix with elements* $\mathfrak{p}_{i,j}$ *defined by equalities*

$$\mathfrak{p}_{i,j} = \widetilde{c}_i\delta_{i,j} \quad \textit{for} \quad i\le k-2,\ j\in\mathbb{Z}, \qquad \mathfrak{p}_{i,j} = \widetilde{c}_{i+1}\delta_{i+1,j} \quad \textit{for} \quad i\ge k+1,\ j\in\mathbb{Z},$$

$$\mathfrak{p}_{k-1,j} = 0 \quad \textit{for} \quad j\in\mathbb{Z}\backslash\{k-1,k\}, \quad \mathfrak{p}_{k-1,k-1} = \widetilde{c}_{k-1},$$

$$\mathfrak{p}_{k-1,k} = \widetilde{c}_{k-1}\frac{\det(\mathbf{a}_k,\ \mathbf{a}_{k+1})}{\det(\mathbf{a}_{k-1},\ \mathbf{a}_{k+1})}, \quad \mathfrak{p}_{k,j} = 0 \quad \textit{for} \quad j\in\mathbb{Z}\backslash\{k,k+1\},$$

$$\mathfrak{p}_{k,k} = \widetilde{c}_{k+1}\frac{\det(\mathbf{a}_{k-1},\mathbf{a}_k)}{\det(\mathbf{a}_{k-1},\ \mathbf{a}_{k+1})}, \quad \mathfrak{p}_{k,k+1} = \widetilde{c}_{k+1}.$$

Matrix $\mathfrak{P}_{(k+1)}$ is called *embedding matrix.*

Let $\{\widetilde{g}_i\}_{i\in\mathbb{Z}}$ be a prolongation onto $\mathbb{S}$ of biorthogonal system to the system $\{\widetilde{\omega}_j\}_{j\in\mathbb{Z}}$ with property $\operatorname{supp}\widetilde{g}_i \in [\widetilde{x}_i, \widetilde{x}_i+\varepsilon)$ $\forall\varepsilon > 0$. If $\varphi \in C^1(\alpha,\beta)$ and $|\det(\varphi,\varphi')(t)| > 0$ $\forall t \in (\alpha,\beta)$, then the mentioned prolongation exists.

By definition, we put $\mathfrak{q}_{i,j} \stackrel{\text{def}}{=} \langle \widetilde{g}_i, \omega_j\rangle$, $\mathfrak{Q}_{(k+1)} \stackrel{\text{def}}{=} (\mathfrak{q}_{i,j})_{i,j\in\mathbb{Z}}$. The matrix $\mathfrak{Q}_{(k+1)}$ is called *prolongation matrix.*

**Theorem 4.** *The relations*

$$\mathfrak{q}_{i,j} = \delta_{i,j}/\widetilde{c}_j \qquad \textit{for } j \le k-1 \quad \forall i \in \mathbb{Z}, \qquad \mathfrak{q}_{k,j} = 0 \quad \forall i \in \mathbb{Z}\backslash\{k-1,k\},$$

$$\mathfrak{q}_{k,k-1} = \frac{1}{\widetilde{c}_{k+1}} \frac{\det(\mathbf{a}_{k+1},\mathbf{a}_k)}{\det(\mathbf{a}_{k-1},\mathbf{a}_k)}, \quad \mathfrak{q}_{k,k} = \frac{1}{\widetilde{c}_{k+1}} \frac{\det(\mathbf{a}_{k-1},\mathbf{a}_{k+1})}{\det(\mathbf{a}_{k-1},\mathbf{a}_k)},$$

$$\mathfrak{q}_{i,k+1} = 0 \quad \forall i \in \mathbb{Z}, \quad \mathfrak{q}_{i,j} = \delta_{i+1,j}/\widetilde{c}_j \qquad \textit{for } j \ge k+2 \quad \forall i \in \mathbb{Z}$$

*are true.*

**Theorem 5.** *Matrix $\mathfrak{Q}_{(k+1)}$ is left inverse matrix to the matrix $\mathfrak{P}^T_{(k+1)}$,*

$$\mathfrak{Q}_{(k+1)}\mathfrak{P}^T_{(k+1)} = \mathfrak{I}, \tag{2}$$

*where $\mathfrak{I}$ is identity matrix.*

Define an operator $P_{(k+1)}$ of projection of the space $\mathbb{S}$ on the subspace $\widetilde{\mathbb{S}}_{k+1}$ by relations

$$P_{(k+1)}u \stackrel{\text{def}}{=} \sum_j \langle \widetilde{g}_j, u\rangle\ \widetilde{\omega}_j \quad \forall u \in \mathbb{S}.$$

By definition, put $Q_{(k+1)} = I - P_{(k+1)}$, where $I$ is identity operator in $\mathbb{S}$.

The space $\mathbb{W}_{k+1} \stackrel{\text{def}}{=} Q_{(k+1)}\mathbb{S}$ is called *the wavelet space.*

Thus we have the direct expansion

$$\mathbb{S} = \widetilde{\mathbb{S}}_{k+1} \dotplus \mathbb{W}_{k+1}. \tag{3}$$

We get *the formula of reconstruction*:

$$\mathbf{c} = \mathfrak{P}^T_{(k+1)}\mathbf{a} + \mathbf{b}, \tag{4}$$

where

$$\mathbf{a} \stackrel{\text{def}}{=} (\dots, a_{-1}, a_0, a_1, \dots)^T, \mathbf{b} \stackrel{\text{def}}{=} (\dots, b_{-1}, b_0, b_1, \dots)^T, \mathbf{c} \stackrel{\text{def}}{=} (\dots, c_{-1}, c_0, c_1, \dots)^T.$$

Vector $\mathbf{c}$ is called *an origin flow*, vector $\mathbf{a}$ is called *a main flow*, vector $\mathbf{b}$ is called *a wavelet flow.* Multiplying both sides of (4) by matrix $\mathfrak{Q}_{(k+1)}$ and using the property (2), we obtain *the formulas of decomposition*:

$$\mathbf{a} = \mathfrak{Q}_{(k+1)}\mathbf{c}, \qquad \mathbf{b} = \mathbf{c} - \mathfrak{P}^T_{(k+1)}\mathfrak{Q}_{(k+1)}\mathbf{c}.$$

**Theorem 6.** *Formulas of reconstruction for spline-wavelet expansion (3) are*

$$c_j = \widetilde{c}_j a_j + b_j \qquad \textit{for} \quad j \le k-1,$$

$$c_k = \widetilde{c}_{k-1} \frac{\det(\mathbf{a}_k,\ \mathbf{a}_{k+1})}{\det(\mathbf{a}_{k-1},\ \mathbf{a}_{k+1})} a_{k-1} + \widetilde{c}_{k+1} \frac{\det(\mathbf{a}_{k-1}, \mathbf{a}_k)}{\det(\mathbf{a}_{k-1},\ \mathbf{a}_{k+1})} a_k + b_k,$$

$$c_j = \widetilde{c}_j a_{j-1} + b_j \qquad \textit{for} \quad j \ge k+1.$$

**Theorem 7.** *Decomposition formulas for expansion (3) are*

$$a_j = c_j / \widetilde{c}_j \qquad \textit{for} \quad j \le k-1,$$

$$a_k = \frac{1}{\widetilde{c}_{k+1}} \Big( \det(\mathbf{a}_{k+1}, \mathbf{a}_k) c_{k-1} + \det(\mathbf{a}_{k-1}, \mathbf{a}_{k+1}) c_k \Big) / \det(\mathbf{a}_{k-1}, \mathbf{a}_k),$$

$$a_j = c_{j+1} / \widetilde{c}_{j+1} \qquad \textit{for} \quad j \ge k+1, \quad b_j = 0 \qquad \forall j \in \mathbb{Z} \backslash \{k+1\},$$

$$b_{k+1} = c_{k+1} - \frac{\det(\mathbf{a}_{k+1}, \mathbf{a}_k)}{\det(\mathbf{a}_{k-1}, \mathbf{a}_k)} c_{k-1} - \frac{\det(\mathbf{a}_{k-1}, \mathbf{a}_{k+1})}{\det(\mathbf{a}_{k-1}, \mathbf{a}_k)} c_k.$$

In previous part of the paper we discuss the wavelet decomposition connecting with removal of one knot. Now we can delete a sequence of knots; in such case we have the next assertion.

**Theorem 8.** *The discussed spline-wavelet expansion doesn't depend on the order of removal of the knots.*

Literature


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