On flat bundles in characteristic 0 and $p > 0$

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- (group of finite type) $\to$ ($\mathbb{C}$-pro-algebraic group) $\to$ (topological pro-finite group)

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- $K = \bar{K} \subset \mathbb{C}$: $\pi_1^{\mathrm{alg,rs}}(X,a) \xrightarrow{\text{pro}-\text{finite completion}} \pi_1^{\text{ét}}(X,a)$ $\pi_1^{\mathrm{alg,rs}}(X \otimes K', a \otimes K') \to \pi_1^{\mathrm{alg,rs}}(X,a) \otimes K'$ not injective (no base change) , $\pi_1^{\text{ét}}(X \otimes K', a \otimes K') = \pi_1^{\text{ét}}(X,a)$ (base change), $K' = \bar{K}' \subset \mathbb{C}$

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- **Thm** (Grothendieck-Malcev over $\mathbb{C}$) Let $f : (X, a) \to (Y, b)$ inducing $\pi_1^{\mathrm{ét}}(X, a) \xrightarrow{f_* \ \cong} \pi_1^{\mathrm{ét}}(Y, b)$. Then $\pi_1^{\mathrm{alg,rs}}(X, a) \xrightarrow{f_* \ \cong} \pi_1^{\mathrm{alg,rs}}(Y, b)$

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- **Thm** (Grothendieck-Malcev over $\mathbb{C}$) Let $f:(X,a)\to(Y,b)$ inducing $\pi_1^{\mathrm{\acute{e}t}}(X,a)\xrightarrow{f_*\ \cong}\pi_1^{\mathrm{\acute{e}t}}(Y,b)$. Then $\pi_1^{\mathrm{alg,rs}}(X,a)\xrightarrow{f_*\ \cong}\pi_1^{\mathrm{alg,rs}}(Y,b)$
- *Proof*: reduce to $\mathbb{C}$. Then algebraic representation $\pi_1^{\mathrm{alg,rs}}(X,a)\to GL(r,\mathbb{C})$ uniquely determined by abstract representation $\rho:\pi_1^{\mathrm{top}}(X,a)\to GL(r,\mathbb{C})$. $\pi_1^{\mathrm{top}}(X,a)$ finitely generated implies ρ has values in $GL(r,A)$, A ring of finite type over $\mathbb{Z}[1/N]$. Reduce ρ to $\rho\otimes\kappa(s):\pi_1^{\mathrm{top}}(X,a)\to GL(r,\kappa(s))$. Then ρ non-trivial iff $\rho\otimes\kappa(s)$ non-trivial for some s.

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- Ex. (ii): X projective, f =Albanese, then $\pi_1^{\text{ét}}(X, a)$ abelian iff irreducible connections have rank 1

Characteristic $p > 0$: Katz' definition

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- so: Tannaka group of a stratification is finite implies it is smooth and a finite étale cover yields such a stratification by push-down of the trivial one

Regular Singular, Tameness

- **Defn** (SGA1): If $X \subset \hat{X}$ is a NCD compactification, then a finite étale cover $Y \to X$ is *tame* iff the finite étale cover of Dedekind schemes at all codim. 1 points of $\hat{X} \setminus X$ is tame.

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- **Defn** (Kindler): E on X is *rs* iff for all $X \subset \hat{X}$ partial (i.e. not necessarily proper) NCD compactifications, E admits a coherent extension $\hat{E}$ to $\hat{X}$ on which $\mathcal{D}_{\hat{X}}(\log(\hat{X} \setminus X))$ acts, extending the action of $\mathcal{D}_X$ on E.

- **Thm** (Kindler):
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- **Cor** (Kindler): E with finite Tannaka group, then E rs iff it is on all $C \to X$.

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- **Thm** (Katz: Ersatz for Riemann-Hilbert): $K = \bar{K}$. Equivalence of Tannaka categories ($\mathcal{O}_X$ coherent $\mathcal{D}_X$ module E) $\leftrightsquigarrow$ (stratifications $(E^n, \sigma^n), E^n$ vector bundle on $X^{(n)}$, $\sigma^n : E^n \cong F^*E^{n+1}$), via Cartier iso. for p-curvature 0 connections

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- **Conj** (Gieseker '75): Ex. (i) in char. > 0, i.e.: X smooth projective over $K = \bar{K}$, if $\pi_1^{\text{ét}}(X, a) = \{1\}$, then no non-trivial stratification?

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- char. $p > 0$: Tannaka formalism allows one to reduce to the case where all the $E^{(n)}$ are stable of degree 0, so are moduli points of quasi-projective moduli M (Langer)

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- $\mathbb{C}$: $\rho \otimes \kappa(s)$, $s \in \operatorname{Spec} A$, controls non-triviality of ρ (trivially)

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- Analog Ex.(ii) also true (second Gieseker conjecture) **Thm** (Esnault-Sun '11): X smooth proejctive over $K = \bar{K}$, then $[\pi_1^{\text{ét}}(X, a), \pi_1^{\text{ét}}(X, a)]$ has no p-power order quotient iff irreducible stratifications have rank 1, and the category of stratifications is semi-simple with irreducible objects of rank 1 iff $\pi_1^{\text{ét}}(X, a)$ is abelian with no p-power order quotient.

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- **Prop** (Esnault-Kindler): (b) OK for $\mathbb{A}^n$