Stability in geometric & functional inequalities

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The stability issue

Geometric and functional inequalities play a crucial role in several problems arising in the calculus of variations, partial differential equations, geometry, etc.
More recently, there has been a growing interest in studying the stability for such inequalities. The basic question one wants to address is the following:
Suppose we are given a functional inequality for which minimizers are known. Can we prove, in some quantitative way, that if a function "almost attains the equality" then it is close (in some suitable sense) to one of the minimizers?

Several results have been obtained in this direction, showing stability for isoperimetric inequalities, the Brunn-Minkowski inequality on convex sets, Sobolev and Gagliardo-Nirenberg inequalities, etc.

The aim of this talk is to describe some ways to attack this kind of problems, and show some applications.

Overview of the talk

1. Stability for isoperimetric inequalities
2. Stability for Gagliardo-Nirenberg and Log-HLS, and long-time behavior for the critical mass Keller-Segel equation

Stability for isoperimetric inequalities

Classical isoperimetric inequality.

For any bounded open smooth set $E \subset \mathbb{R}^n$, the perimeter $P(E)$ controls the volume $|E|$:

$$P(E) \geq n|B_1|^{1/n}|E|^{(n-1)/n}.$$

Moreover equality holds if and only if E is a ball.

Stability question: if E is "almost a minimizer" does this imply that E is close to a ball, if possible in some quantitative way?

Isoperimetric deficit of E:

$$\delta(E) := \frac{P(E)}{n|B_1|^{1/n}|E|^{(n-1)/n}} - 1\,.$$

Observe that $\delta(E) \geq 0$.
Moreover $\delta(E) = 0$ if and only if E is a ball.

Asymmetry index of E:

$$A(E) := \inf_{x,r}\left\{\frac{|E\Delta(B_r(x))|}{|E|} : |B_r| = |E|\right\}$$

Here $E\Delta F$ denotes the symmetric difference between the sets E and F, i.e., $E\Delta F := (E \setminus F) \cup (F \setminus E)$.

Question: can we find positive constants $C = C(n)$ and $\alpha = \alpha(n)$ such that

$$A(E) \leq C\,\delta(E)^{\alpha}\,?$$

Remark: by testing the above inequality on a sequence of ellipsoids converging to B_1, we get $\alpha \leq 1/2$.

This is actually the sharp result:

Theorem (Fusco-Maggi-Pratelli, 2008)

The stability result holds with $\alpha = 1/2$*.*

The proof of Fusco-Maggi-Pratelli uses symmetrization techniques which are very specific to the Euclidean case.

We now describe a different approach which has the advantage to work for much more general perimeter-type functionals. More precisely, we replace the classical perimeter by

$$P_f(E) := \int_{\partial E} f(\nu_E)$$

with f positively 1-homogeneous and convex, and we look for the corresponding isoperimetric inequality (the so-called *Wulff inequality*).

Theorem (Figalli-Maggi-Pratelli, 2010)

The stability result still holds for P_f with $\alpha = 1/2$, and C an explicit constant depending only on the dimension (and not on f).

Gromov's proof of the isoperimetric inequality

Given E smooth and bounded, consider the probability measures

$$\mu := \frac{\chi_E(x)}{|E|}\,dx, \qquad \nu := \frac{\chi_{B_1}(y)}{|B_1|}\,dy.$$

By optimal transport theory, there exists $\varphi : \mathbb{R}^n \to \mathbb{R}$ convex such that $T := \nabla\varphi$ sends μ onto ν:

$$T_\#\mu = \nu \qquad \text{(i.e. } \mu(T^{-1}(A)) = \nu(A) \text{ for all } A \text{ Borel)}$$

Properties of T:

1. $|T| \leq 1$ in E (since $T(E) \subset B_1$)
2. $\det(DT) = |B_1|/|E|$ (since $T_{\#}\mu = \nu$)
3. $\operatorname{div} T \geq n(\det(DT))^{1/n}$ (wait for the next slide).

Then:

$$\begin{aligned} P(E) &= \int_{\partial E} 1 \overset{(1)}{\geq} \int_{\partial E} |T| \geq \int_{\partial E} T \cdot \nu_E \\ &= \int_E \operatorname{div} T \overset{(3)}{\geq} n \int_E (\det(DT))^{1/n} \\ &\overset{(2)}{=} n|B_1|^{1/n}|E|^{(n-1)/n}. \end{aligned}$$

Let's prove (3):
since $T = \nabla\varphi$ with φ convex, the eigenvalues $\lambda_1, \dots, \lambda_n$ of $D^2\varphi$ are non-negative.
Hence:

$$\operatorname{div} T = \Delta\varphi = n\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right) \geq n\left(\prod_{i=1}^{n}\lambda_i\right)^{1/n} = n(\det(DT))^{1/n},$$

where we used the arithmetic-geometric inequality.

This proof works also for the functional P_f and is very robust.
In particular, by carefully making "quantitative" each inequality one can prove the desired stability result.

Long-time asymptotic for the critical mass Keller-Segel equation

The Keller-Segel equation describes the evolution of a cell population ρ under the influence of a chemical attractant c produced by the cells themselves.

Then the cell flux comprises two counteracting phenomena: random motion of the cells described by Fick's law (diffusion), and a tendency to move towards higher concentrations of the attractant (drift).

The *Keller-Segel system*:

$$\frac{\partial \rho}{\partial t}(t,x) = \Delta \rho(t,x) - \operatorname{div}\big[\rho(t,x)\nabla c(t,x)\big] \,.$$

Here $\rho(0,x) \in L^1(\mathbb{R}^2)$ is non-negative, and c satisfies $-\Delta c = \rho$, that is

$$c(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\rho(t,y)\,dy \,.$$

Formal conservation laws:

$$\int_{\mathbb{R}^2} \rho(t,x)\,dx = \int_{\mathbb{R}^2} \rho(0,x)\,dx =: M,$$

$$\frac{d}{dt}\int_{\mathbb{R}^2} x\rho(t,x)\,dx = 0,$$

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\rho(t,x)\,dx = 4M - \frac{1}{2\pi}M^2$$

Since $\int |x|^2 \rho \geq 0$, something has to "go wrong" if $M > 8\pi$.

Indeed, it is by now well-known that:

1. $M < 8\pi$: diffusion dominates and the solution diffuses away to infinity.
2. $M > 8\pi$: the restoring drift dominates and the solution collapses in finite time.
3. $M = 8\pi$ (critical mass case): solution exists globally in time and there are infinitely many steady-states, which (up to a translation) are given by

$$\sigma_\kappa(x) := \frac{8\kappa}{\left(\kappa + |x|^2\right)^2}, \qquad \kappa > 0.$$

Assume from now on $M = 8\pi$ and $\int x\rho(0,x)\,dx = 0$.

Question: if $\rho(t,\cdot) \to \sigma_\kappa$ as $t \to \infty$, how to select κ?

Answer: use the energy functionals

$$\mathcal{H}_\kappa[\rho] := \int_{\mathbb{R}^2} \frac{|\sqrt{\rho}(y) - \sqrt{\sigma_\kappa}(y)|^2}{\sqrt{\sigma_\kappa}(y)}\,dy\,.$$

These functionals are decreasing along KS.

Moreover $\mathcal{H}_\kappa[\sigma_{\kappa'}] = \infty$ unless $\kappa' = \kappa$.

So, if $\mathcal{H}_{\kappa_0}[\rho(0,\cdot)] := E_0 < \infty$ then $\mathcal{H}_{\kappa_0}[\rho(t,\cdot)] \le E_0$, and $\rho(t,\cdot)$ should converge to σ_{κ_0} as $t \to \infty$.

This has been proved by Blanchet-Carrillo-Carlen (2010) using a compactness argument. Our goal is to obtain an explicit rate of convergence.

Strategy:
differentiating $\mathcal{H}_{\kappa_0}$ along KS we get

$$\frac{d}{dt}\mathcal{H}_{\kappa_0}[\rho(t,\cdot)] = -\mathcal{D}[\rho(t,\cdot)]\,,$$

with

$$\mathcal{D}[\sigma] := \frac{1}{\pi}\left(\|\nabla u\|_2^2\|u\|_4^4 - \pi\|u\|_6^6\right)\,, \qquad u := \sigma^{1/4}.$$

Note: by the Gagliardo-Nirenberg inequality (Del Pino-Dolbeault, 2002),

$$\mathcal{D}[\sigma] \geq 0.$$

In addition equality holds if and only if σ is a multiple of $\sigma_\kappa(\cdot - x_0)$ for some $x_0 \in \mathbb{R}^2$, $\kappa > 0$.

Question: *When* $\mathcal{D}[\sigma]$ *is small, is* σ *close (in some sense) to a multiple of* $\sigma_\kappa(\cdot - x_0)$*?*

Theorem (Carlen-Figalli, 2011)

Let $\sigma \geq 0$, $\|\sigma\|_1 = 8\pi$*. Then*

$$\inf_{\kappa>0, x_0\in\mathbb{R}^2} \|\sigma^{3/2} - \sigma_\kappa(\cdot - x_0)^{3/2}\|_1 \lesssim \mathcal{D}[\sigma]^{1/2}.$$

Integrating in time the relation

$$\frac{d}{dt}\mathcal{H}_{\kappa_0}[\rho(t,\cdot)] = -\mathcal{D}[\rho(t,\cdot)]\,,$$

we get (recall that $\mathcal{H}_{\kappa_0} \geq 0$)

$$\inf_{t\in[0,T]} \mathcal{D}[\rho(t,\cdot)] \leq \frac{1}{T}\int_0^T \mathcal{D}[\rho(t,\cdot)]\,dt \leq \frac{1}{T}\mathcal{H}_{\kappa_0}[\rho(0,\cdot)]\,.$$

Hence, by the stability result for GN

$$\|\rho(\bar{t},\cdot)^{3/2} - \sigma_\kappa(\cdot - x_0)^{3/2}\|_1 \leq \frac{C}{\sqrt{T}}$$

for some $\bar{t} \in [0,T]$, $x_0 \in \mathbb{R}^2$, $\kappa > 0$.
Using that the baricenter is preserved in time, we easily get rid of x_0.
Moreover, by some interpolation argument, we have

$$\|\rho(\bar{t},\cdot) - \sigma_\kappa\|_1 \leq \frac{C}{T^\alpha}$$

for some $\alpha > 0$, $\kappa = \kappa(\bar{t})$.

Problems:

1. Show that $\rho(t, \cdot)$ approaches σ_κ for $\kappa = \kappa_0$.
2. Show that eventually it *remains* close.

While (1) is easier, (2) requires much more work.

Additional tool: exploit that the Logarithmic Hardy-Littlewood-Sobolev (Log-HLS) functional

$$\mathcal{F}[\rho] := \int_{\mathbb{R}^2} \rho(x) \log \rho(x)\, dx + 2 \left(\int_{\mathbb{R}^2} \rho(x)\, dx \right)^{-1} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x) \log |x - y| \rho(y)\, dx\, dy$$

is decreasing along KS, and is uniquely minimized at $\{\sigma_\kappa\}_{\kappa>0}$.
We prove a "two sided stability" for it:

Theorem (Carlen-Figalli, 2011)

$$\inf_\kappa \|\rho - \sigma_\kappa\|_1^{\beta_1} \lesssim \mathcal{F}[\rho] - \min \mathcal{F} \lesssim \inf_\kappa \|\rho - \sigma_\kappa\|_1^{\beta_2}.$$

Hence, since $\mathcal{F}$ is decreasing in time:

$$\frac{C}{T^\alpha} \geq \inf_\kappa \|\rho(\bar{t},\cdot) - \sigma_\kappa\|_1 \gtrsim \big(\mathcal{F}[\rho(\bar{t},\cdot)] - \min \mathcal{F}\big)^{1/\beta_2}$$
$$\geq (\mathcal{F}[\rho(T,\cdot)] - \min \mathcal{F})^{1/\beta_2} \gtrsim \inf_\kappa \|\rho(T,\cdot) - \sigma_\kappa\|_1^{\beta_1/\beta_2}$$

for *all* $T \gg 1$.

Finally $\mathcal{H}_{\kappa_0}[\rho(T,\cdot)] \leq E_0$ implies that the infimum above is attained at $\kappa(T)$, with

$$|\kappa(T) - \kappa_0| \leq \frac{C}{\sqrt{\log(1+T)}}.$$

This allows to prove a “two-scale” convergence result:

Theorem (Carlen-Figalli, 2011)

It holds:

$$\inf_{\kappa>0} \|\rho(t,\cdot) - \sigma_\kappa\|_1 \leq C\,(1+t)^{-(1-\epsilon)/320}.$$

Moreover, the above infimum is achieved at some value $\kappa(t)$ *satisfying*

$$|\kappa(t) - \kappa_0| \leq \frac{C}{\sqrt{\log(e+t)}}.$$

In particular

$$\|\rho(t,\cdot) - \sigma_{\kappa_0}\|_1 \leq \frac{C}{\sqrt{\log(e+t)}}.$$

It is interesting that the approach to equilibrium described by these quantitative bounds takes place on two separate time scales: The solution approaches the one-parameter family of (centered) stationary states with at least a polynomial rate.
Then, perhaps much more gradually, at only a logarithmic rate, the solution adjusts its spatial scale to finally converge to the unique stationary solution within its basis of attraction.

It looks reasonable to expect such behavior: The initial data may, for example, be exactly equal to σ_{κ_0} on the complement of a ball of very large radius R, and yet may "look much more like" σ_κ on a ball of smaller radius for some $\kappa \neq \kappa_0$. One can then expect the solution to first approach σ_κ, and then only slowly begin to feel its distant tails and make the necessary adjustments to the spatial scale.

THAT'S ALL!!

Thanks for your attention!