Remarks on the global regularity for solutions to the incompressible Navier-Stokes equations

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Presentation of the equations

- Viscous, incompressible, homogeneous fluid, in $\mathbb{R}^3$
- Velocity $u = (u^1, u^2, u^3)(t,x)$, pressure $p(t,x)$

$$(NS) \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \\ \operatorname{div} u = 0 \end{cases}$$

with

$$\Delta u = \sum_{j=1}^{3} \partial_j^2 u \,, \qquad \operatorname{div} u = \sum_{j=1}^{3} \partial_j u^j \,, \qquad \partial_j := \frac{\partial}{\partial x_j} \;, \quad \partial_t := \frac{\partial}{\partial t}$$

$$u \cdot \nabla u = \sum_{j=1}^{3} u^j \partial_j u = \sum_{j=1}^{3} \partial_j (u^j u) \,.$$

Remark : The pressure can be eliminated by **projection onto divergence-free vector fields** : $\mathbb{P} = \mathrm{Id} - \nabla \Delta^{-1} \mathrm{div}$.

Cauchy data : $u_{|t=0} = u_0$.

Solving the equations

We want to find $u(t,x)$ solution to (NS) in some sense (distributional, classical...), such that $u(0,x) \equiv u_0(x)$.

Standard methods :

- **Compactness** methods :

- Find an *a priori* bound on the solution : $\|u(t)\|_X \leq C(u_0)$;

- Construct a sequence of *approximate equations* $(\mathrm{NS})_n$ which can be solved by the Cauchy-Lipschitz theorem : this yields a sequence of *approximate solutions* $(u_n)_{n\in\mathbb{N}}$, uniformly bounded in X ;

- Use the uniform bound in X to construct weak limit points to the sequence $(u_n)_{n\in\mathbb{N}}$: $u_n \rightharpoonup u$;

- Use space-time compactness to prove that u solves (NS).

Solving the equations

- Banach **fixed point** theorem :

- Write the equation in integral form :

$$u(t) = e^{t\Delta}u(0) + B(u,u)(t)$$

- Apply a fixed point theorem.

Fundamental properties of (NS)

- **Conservation of the energy**

Define

$$\|f\|_{L^2} := \Big(\int_{\mathbb{R}^3} |f(x)|^2 \, dx \Big)^{\frac{1}{2}} .$$

Then conservation of energy is due to the formal identity

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2 \, dt' = \frac{1}{2}\|u_0\|_{L^2}^2$$

thanks to the structure of the nonlinear term : $\big(\mathbb{P}(u \cdot \nabla u)|u\big)_{L^2} = 0$. So in particular $u \in L^\infty(\mathbb{R}^+; L^2)$ and $\nabla u \in L^2(\mathbb{R}^+; L^2)$.

Fundamental properties of (NS)

- **Scale invariance**

If $u(t,x)$ is a solution of (NS) associated with the initial data $u_0(x)$ on $[0,T]\times\mathbb{R}^3$, then for all $\lambda>0$, $a\in\mathbb{R}^3$

$$u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda(x-a))$$

is a solution associated with $u_{\lambda,0}(x) := \lambda u_0(\lambda(x-a))$ on $[0,\lambda^{-2}T]\times\mathbb{R}^3$.

Weak solutions

Using the **conservation of energy**, one can prove the following result.

Theorem [Leray, 1934]

Let $u_0 \in L^2(\mathbb{R}^d)$ be a divergence free vector field. There is a solution u of (NS) satisfying for all $t \geq 0$

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2}^2 \, dt' \leq \|u_0\|_{L^2}^2 \, .$$

Remarks :

- Proof by compactness.
- Search for conditions on the initial data to guarantee uniqueness (if $d = 2$, OK due to scale invariance).

Strong solutions

One does not use the **structure** of the equation, but rather its **scale invariance**, by a **fixed point** method.

Solving (NS) is equivalent to solving

$$u = e^{t\Delta} u_0 + \mathbb{B}(u, u)$$

where $e^{t\Delta}$ is the heat semi-group on $\mathbb{R}^d$ and $\mathbb{B}$ the bilinear form

$$\mathbb{B}(u, u)(t) := -\int_0^t e^{(t-t')\Delta} \mathbb{P}\,\mathrm{div}\,(u \otimes u)(t')\, dt'\,.$$

The problem consists in finding an **adapted** Banach space X, such that $\mathbb{B}$ is continuous from $X \times X$ to X.

An existence and uniqueness result

Theorem

Let X be an adapted space. If u_0 is such that $\|e^{t\Delta}u_0\|_X$ is small enough, then there is a unique solution to (NS) in X.

Proof : This is simply Picard's theorem : if X is a Banach space and $\mathbb{B} \in \mathcal{B}(X)$, then for all x_0 in X satisfying $\|x_0\|_X < \dfrac{1}{4\|\mathbb{B}\|_{\mathcal{B}(X)}}$ the equation

$$x = x_0 + \mathbb{B}(x,x)$$

has a unique solution in the ball centered at 0 and of radius $\dfrac{1}{2\|\mathbb{B}\|_{\mathcal{B}(X)}}\cdot$

Remark : By scale invariance, the norm on X must satisfy

$$\forall \lambda > 0\,, \forall x \in \mathbb{R}^3\,, \quad \lambda \| f(\lambda^2 t, \lambda(x-a))\|_X \sim \|f\|_X$$

The optimal adapted space

Define

$$\|u_0\|_{B_p} := \sup_{t>0} \; t^{\frac{1}{2}(1-\frac{d}{p})}\|e^{t\Delta}u_0\|_{L^p} \,.$$

- Any Banach space of tempered distributions, scale and translation invariant, is **embedded in** B_∞ [Meyer '96]
- (NS) is **ill-posed** in B_∞ [Bourgain-Pavlovic '08, Germain '08]
- (NS) is **well-posed** (for small enough data) in $\widetilde{B}_\infty$ where

$$\|u_0\|_{\widetilde{B}_\infty} := \sup_{t\geq 0} t^{\frac{1}{2}}\|e^{t\Delta}u_0\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} \frac{1}{R^{\frac{3}{2}}}\Big(\int_{P(x,R)} |(e^{t\Delta}u_0)(t,y)|^2 dy\Big)^{\frac{1}{2}}$$

[Koch-Tataru '01]

Remarks

• Previous results in this framework are due to Leray '34 (smallness measured by $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ if $d=3$), Fujita-Kato '64 (with $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}$), Kato '84 (with $\|u_0\|_{L^d}$), Cannone-Meyer-Planchon '94 (with $\|u_0\|_{B_p}$).

• In this context in general, only **small data** or **small time** theorems are known. They hold for the more general equation

$$\partial_t u - \Delta u = Q(u,u)$$

where $Q(v,w) := \sum_{1\leq j,k\leq 3} Q_{j,k}(D)(v^j w^k)$ and $Q_{j,k}(D)$ are smooth homogeneous Fourier multipliers of order 1.
However some of these equations are **known to blow-up** [Montgomery-Smith '01], including for (large) data for which Navier-Stokes does not [G-Paicu '09].

Remarks

• Unfortunately there is a **discrepancy** between the energy (providing control of norms) and the scaling (necessary to implement the fixed point).

If $d = 2$, the energy space scale invariant, the equation is said **critical**. In dimension $d \geq 3$, there are $d/2 - 1$ derivatives between scaling and energy : the equation is said **supercritical**.

Properties of $\mathcal{G}$

In the following we denote by $\mathcal{G}$ the space of initial data generating a **global smooth solution** to the three-dimensional Navier-Stokes equations.

We want to study **geometrical** properties of $\mathcal{G}$.
We shall prove that $\mathcal{G}$ is

- **open (strong topology)** [G-Iftimie-Planchon '03]
- **connected** in $\dot{H}^{\frac{1}{2}}$, B_p [G-Iftimie-Planchon '03], BMO^{-1} [Auscher, Dubois, Tchamitchian '04]
- **unbounded** in B_∞ [Chemin-G '06,'09,'10, Chemin-G-Paicu '12, Chemin-G-Zhang '12]
- **open (weak topology)** (under an anisotropy assumption) [Bahouri-G '12, Bahouri-Chemin-G in progress].

The set $\mathcal{G}$ is **strongly** open, and connected

Let us prove the following result.

Theorem [G-Iftimie-Planchon '03]

Let $u \in C(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ be a solution to (NS). Then

$$\lim_{t \to +\infty} \|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = 0\,.$$

Moreover u is stable in the sense that there is $\varepsilon > 0$ such that if $\|u_{|t=0} - v_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \varepsilon$ then there is a **unique global solution** associated with v_0.

Remarks.

- The same result holds in the more general framework of BMO^{-1} [Auscher, Dubois, Tchamitchian '04].

- The result shows that $\mathcal{G}$ is **open** in the strong topology. An immediate corollary of the theorem is that $\mathcal{G}$ is **connected**.

Idea of the proof of the result (large time behaviour)

An easy case : assume $u_0 := u_{|t=0} \in L^2 \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then u satisfies the energy inequality and in particular $u \in L^4(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ so there is t_0 such that $\|u(t_0)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \varepsilon_0$ and then that holds for all $t \geq t_0$ by small data theory.

The general case : write $u_0 = v_0 + w_0$ with w_0 small in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and v_0 in $L^2(\mathbb{R}^3)$.

Solve (NS) globally with the data w_0, the solution $w(t)$ remains small in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for all times.

Prove that the solution of

$$\partial_t v + \mathbb{P}(-v \cdot \nabla v + v \cdot \nabla w + w \cdot \nabla v) - \Delta v = 0$$

is bounded in the energy space and conclude as above.

The set $\mathcal{G}$ is **weakly** open

We consider sequences **converging weakly** to an element of $\mathcal{G}$.

Examples : the sequence $\phi_n(x) := 2^n\phi(2^n x)$ converges weakly to zero. If $\mathcal{G}$ were open for the weak topology then ϕ would belong to $\mathcal{G}$ by scale invariance... The same goes for $\widetilde{\phi}_n(x) := \phi(x - x_n)$, $|x_n| \to \infty$.

Define Δ_k^h and Δ_j^v Littlewood-Paley frequency truncation operators :

$$\mathcal{F}(\Delta_k^h f)(\xi) := \varphi(2^{-k}\xi_1, 2^{-k}\xi_2, \xi_3)\mathcal{F}(f)(\xi)\,,$$

$$\mathcal{F}(\Delta_j^v f)(\xi) := \varphi(\xi_1, \xi_2, 2^{-j}\xi_3)\mathcal{F}(f)(\xi)$$

where $\varphi \in \mathcal{C}_c^\infty(\frac{1}{2}\,, 1)$. Notice that

$$\|\Delta_k^h \partial_1 f\|_{L^p} \sim 2^k \|\Delta_k^h f\|_{L^p}\,.$$

Then consider the norm $\|f\|_{\mathcal{B}_q^1} := \Big(\sum_{j,k} 2^{(j+k)q} \|\Delta_k^h \Delta_j^v f\|_{L^1(\mathbb{R}^3)}^q\Big)^{\frac{1}{q}}$.

Remark : scale invariance of (NS).

The set $\mathcal{G}$ is **weakly** open

Definition

Let $0 < q \leq \infty$ be given. We say that a sequence $(f_n)_{n\in\mathbb{N}}$, bounded in $\mathcal{B}^1_q$, is ***anisotropically oscillating*** if the following property holds : for all sequences (k_n, j_n),

$$\limsup_{n\to\infty} 2^{j_n+k_n} \|\Delta^h_{k_n}\Delta^v_{j_n} f_n\|_{L^1} = C > 0 \quad \Longrightarrow \lim_{n\to\infty} |j_n - k_n| = \infty\,.$$

Example : the sequence

$$\phi_n(x) := 2^{\alpha n}\phi(2^{\alpha n}x_1, 2^{\alpha n}x_2, 2^{\beta n}x_3)\,,\ \alpha \neq \beta$$

is anisotropically oscillating : horizontal frequencies $\sim 2^{\alpha n}$ and vertical frequencies $\sim 2^{\beta n}$ so

$$\limsup_{n\to\infty} 2^{j_n+k_n} \|\Delta^h_{k_n}\Delta^v_{j_n} \phi_n\|_{L^1} = C > 0 \quad \Longrightarrow k_n \sim \alpha n\,,\ j_n \sim \beta n\,.$$

The set $\mathcal{G}$ is **weakly** open

Theorem [Bahouri-G '12, Bahouri-Chemin-G in progress]

Let $q \in]0,1[$ be given and let $(u_{0,n})_{n\in\mathbb{N}}$ be a sequence of divergence free vector fields bounded in $\mathcal{B}^1_q$, converging towards $u_0 \in \mathcal{B}^1_q$ in the sense of distributions, with $u_0 \in \mathcal{G}$. If $u_0 - (u_{0,n})_{n\in\mathbb{N}}$ is anisotropically oscillating, then up to extracting a subsequence, $u_{0,n} \in \mathcal{G}$ for all $n \in \mathbb{N}$.

Remarks.

- One can essentially consider **any** bounded sequence **except** for sequences of the type described above and their superpositions.

- The theorem may be generalized by adding two more sequences to $(u_{0,n})_{n\in\mathbb{N}}$, where in each additional sequence the "privileged" direction is not x_3 but x_1 or x_2.

- The same result holds for data not in $\mathcal{G}$, on some life span $[0, T]$ for $T < T^\star$.

The rest of the talk is devoted to a sketch of the proof of this result.

1. Write down an **"anisotropic profile decomposition"** of the sequence of initial data. This allows to replace the sequence of initial data, up to an arbitrarily small remainder term, by a finite (but large) sum of profiles of the type

$$\frac{1}{\lambda_n}\Phi\Big(\frac{x_1}{\lambda_n},\frac{x_2}{\lambda_n},\frac{h_n x_3}{\lambda_n}\Big) \quad h_n \to 0\,.$$

2. Propagate **globally in time** by (NS) each individual profile of the decomposition.

3. Prove that the construction of the previous step does provide, after superposition of all the global solutions, an **approximate solution** to the Navier-Stokes equations.

Before carrying out that program we shall **discuss an example** of the type above.

A (typical) example

Consider the divergence-free initial data

$$\Phi_{0,n}(x) := \frac{1}{\lambda_n}(\Phi_0^1, \Phi_0^2, 0)\Big(\frac{x_1}{\lambda_n}, \frac{x_2}{\lambda_n}, \frac{h_n x_3}{\lambda_n}\Big), \quad h_n \to 0\,.$$

Up to rescaling by λ_n it is equivalent to study

$$\widetilde{\Phi}_{0,n}^h(x) := \Phi_0^h\big(x_h, h_n x_3\big), \quad x_h := (x_1, x_2)\,, \; \Phi_0^h := (\Phi_0^1, \Phi_0^2)\,.$$

A (typical) example

Let

$$\widetilde{\Phi}^h_{0,n}(x) := \Phi^h_0\big(x_h, h_n x_3\big)\,, \quad x_h := (x_1, x_2)\,,\, \Phi^h_0 := (\Phi^1_0, \Phi^2_0)\,.$$

To prove there is a **unique global solution** to (NS) associated with $\widetilde{\Phi}^h_{0,n}$ for n large enough [Chemin-G '10], we start by solving globally the **two dimensional equations** with data $\Phi^h_0(x_h, y_3)$ for each y_3. We denote by $\Phi^h(t, x_h, y_3)$ the solution.

Then we check that $(\Phi^h, 0)(t, x_h, h_n x_3)$ is a global **approximate solution** to (NS) with data $\widetilde{\Phi}_{0,n}$, so by rescaling, a global approximate solution associated with $\Phi_{0,n}$ is

$$\Phi_n(t,x) := \frac{1}{\lambda_n}(\Phi^h, 0)\big(\frac{t}{\lambda_n^2}\,, \frac{x_h}{\lambda_n}\,, \frac{h_n x_3}{\lambda_n}\big)\,.$$

Another (typical) example

In the previous example

$$\Phi_{0,n}(x) := \frac{1}{\lambda_n}(\Phi_0^1, \Phi_0^2, 0)\Big(\frac{x_1}{\lambda_n}, \frac{x_2}{\lambda_n}, \frac{h_n x_3}{\lambda_n}\Big), \quad h_n \to 0,$$

we had $\Phi_{0,n}^h \rightharpoonup 0$ if $\lambda_n \to 0$ or ∞ and $\Phi_{0,n}^h(x) \rightharpoonup \Phi_0^h(x_h, 0)$ if $\lambda_n \equiv 1$.

Consider now the divergence-free initial data

$$u_{0,n} := u_0 + (\Phi_0^h, 0)\big(x_1, x_2, h_n x_3\big),$$

with $u_0 \in \mathcal{G}$. We assume that $u_{0,n} \rightharpoonup u_0$ so $(\Phi_0^h, 0)(x_h, 0) \equiv 0$.

We know there is a global solution to (NS) associated with $\Phi_{0,n}$, denoted Φ_n, and we call u the global solution associated with u_0. We want to prove that $u + \Phi_n$ is a global, approximate solution to (NS).

Another (typical) example

Since $(\Phi_0^h, 0)(x_h, 0) \equiv 0$, then $\Phi_n(t, x_h, 0) \sim 0$ so up to a small error, the support in x_3 of Φ_n is $\sim h_n^{-1} \to \infty$.

Approximating u by a compactly supported vector field we find that the supports of u and Φ_n are asymptotically disjoint, so the two vector fields **do not interact**.

That ends the proof in this model case.