Solving the KPZ equation

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Introduction

Object of study: KPZ equation of surface growth:

$$\partial_t h = \partial_x^2 h + \lambda(\partial_x h)^2 + \xi - \infty$$

with either $x \in \mathbf{R}$ or $x \in S^1$, and ξ is space-time white noise.

1. Universal model for interface fluctuations. (Shown rigorously only for SOS model, see Bertini-Giacomin 1997.)
2. Free energy for polymer models.
3. Scaling limit of time-dependent parabolic Anderson model.
4. Universal object describing crossover from Edwards-Wilkinson to KPZ.

Problem: Right-hand side is badly ill-posed! (Solutions only $\mathcal{C}^\alpha$ for $\alpha < \frac{1}{2}$!!)

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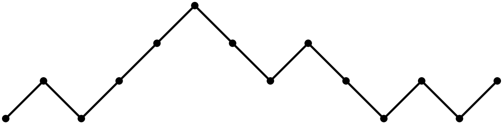
The SOS model

Simplest possible model of surface growth. Surface modelled by graph with slope ± 1:



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Theorem (Bertini & Giacomin, 1997): $\varepsilon^{\frac{1}{2}} h(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) - \varepsilon^{-1}$ converges to KPZ.

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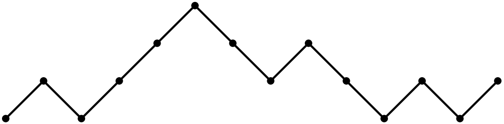

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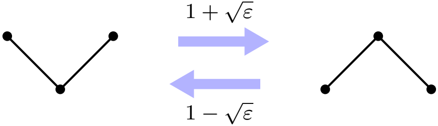

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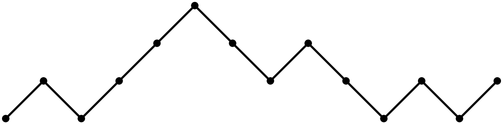

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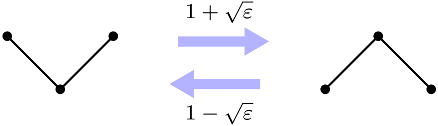

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Cole-Hopf solution

Trick introduced by Cole and Hopf in the 50's. Write

$$h = \lambda^{-1} \log Z \ ,$$

then Z solves

$$\partial_t Z = \partial_x^2 Z + \lambda Z \, \xi \ . \qquad (\star)$$

Idea: Take this as definition of solution, where $(\star)$ is interpreted in the Itô sense. Work by Bertini-Giacomin shows that this is the physically relevant solution.

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Properties of Cole-Hopf

Mollify W, so $W_{\varepsilon,k} = \varphi(\varepsilon k) W_k$ for cutoff φ, and set

$$dZ_\varepsilon = \partial_x^2 Z_\varepsilon \, dt + \lambda Z_\varepsilon \, dW_\varepsilon \,, \quad h_\varepsilon = \lambda^{-1} \log Z_\varepsilon \,.$$

Then h_ε solves

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \lambda\big((\partial_x h_\varepsilon)^2 - C_\varepsilon\big) + \xi_\varepsilon \,, \quad C_\varepsilon \approx \frac{1}{\varepsilon} \int \varphi^2 \,.$$

Problems with this notion of solution:

1. Not satisfactory at the formal level.
2. Lack of robustness: no good approximation theory for other modifications (hyperviscosity, time-smoothing, etc).
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Some attempts

1. Consider product as Wick product $\partial_x h \diamond \partial_x h$ (Øksendal & Al 1995): wrong notion of solution ($\neq \mathcal{S}_{CH}$). Also wrong scaling properties (Chan 2000).

2. Formulate as martingale problem (Assing 2002): no well-posedness, "generator" not shown to be closable.

3. Apply "standard" renormalisation techniques inspired by QFT (Da Prato, Debussche, Tubaro 2007): only works for a regularised equation.

4. Define nonlinearity on some distributional space (Gonçalves, Jara 2010, Assing 2011): no uniqueness. No characterisation of class of distributions for which formulation even makes sense.

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A robustness result

Theorem (H. 2011): For $\alpha > 0$ arbitrary one can build the following objects:

$$\begin{array}{ccccc}
\mathcal{X} & \times & \mathcal{C}^\alpha & \xrightarrow{\mathcal{S}_R} & \mathcal{C}(\mathbf{R}_+, \mathcal{C}^\alpha) \\
\Psi \uparrow & & \uparrow & & \downarrow \\
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\end{array} .$$

where $\mathcal{S}_R$ is jointly continuous, but Ψ is only measurable. (Slight cheat: solutions are only local in general.)

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A deterministic result

Consider solutions h_ε to

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 + \varepsilon^{-3/2} g(\varepsilon^{-1}x - \varepsilon^{-2}t) - K_\varepsilon \ ,$$

for centred periodic g and suitable large constants K_ε. Then, one can compute K such that $h_\varepsilon \to h$ solving

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Proof: Just show that $\Psi(g_\varepsilon)$ converges to a limit in $\mathcal{X}$...

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Ideas of technique

Idea: Perform Wild expansion of solution: define

$$\partial_t Y_\varepsilon^\bullet = \partial_x^2 Y_\varepsilon^\bullet + \xi_\varepsilon \ .$$

For any binary tree $\tau = [\tau_1, \tau_2]$, define Y_ε^τ recursively by

$$\partial_t Y_\varepsilon^\tau = \partial_x^2 Y_\varepsilon^\tau + \partial_x Y_\varepsilon^{\tau_1} \, \partial_x Y_\varepsilon^{\tau_2} - C_\varepsilon^\tau \ .$$

Formal calculation shows that

$$h_\varepsilon(t) = \sum_\tau \lambda^{|\tau|-1} Y_\varepsilon^\tau(t) \ ,$$

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A convergence result

Theorem: For every τ, there is a choice of α_τ and C_ε^τ such that

$$Y_\varepsilon^\tau \to Y^\tau \qquad \text{(Independent of } \varphi\text{.)}$$

in probability in $\mathcal{C}(\mathbf{R}, \mathcal{C}^\alpha) \cap \mathcal{C}^\beta(\mathbf{R}, \mathcal{C})$ for $\alpha < \alpha_\tau$ and $\beta < \frac{1}{2}$.

Optimal choice: $\alpha_\bullet = \frac{1}{2}$, $\alpha_{\vee} = 1$, $\alpha_\tau = (\alpha_{\tau_1} \wedge \alpha_{\tau_2}) + 1$.

$$\begin{aligned}
C_\varepsilon^{\vee} &= C_\varepsilon \sim \frac{1}{\varepsilon} \int_{\mathbf{R}} \varphi^2(x)\, dx \;, \\
C_\varepsilon^{\vee\!\vee} &= \frac{4\pi}{\sqrt{3}} |\log \varepsilon| - C(\varphi) \;, \\
C_\varepsilon^{\vee\!\!\vee} &= -\tfrac{1}{4} C_\varepsilon^{\vee\!\vee} \;.
\end{aligned}$$

Truncated expansion

Idea: Write h_ε as

$$h_\varepsilon = \sum_{\tau \in \mathcal{T}} \lambda^{|\tau|-1} Y_\varepsilon^\tau + u_\varepsilon \ ,$$

for a finite set $\mathcal{T}$, derive an equation for u_ε, and pass to limit. Minimal working choice: $\mathcal{T} = \{\bullet, \RSV, \RSW, \RSVV, \RSWV\}$. One obtains

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + 2\lambda\, \partial_x u_\varepsilon\, \partial_x Y_\varepsilon^\bullet + \text{"l.o.t."} \ .$$

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$$\partial_t u = \partial_x^2 u + 2\lambda\, \partial_x u\, \partial_x Y^\bullet \ .$$

"Theorem:" There exists no pair of Banach spaces containing u and $Y^\bullet$ such that the right-hand side makes sense. (Very different from DiPerna - Lions, closer to Flandoli - Russo.)

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“Theorem:” There exists no pair of Banach spaces containing u and $Y^\bullet$ such that the right-hand side makes sense. (Very different from DiPerna - Lions, closer to Flandoli - Russo.)

How to solve that equation?

Writing $v = \partial_x u$, recall we want to solve

$$\partial_t v = \partial_x^2 v + 2\lambda\, \partial_x \big(v\, \partial_x Y^\bullet\big)\ .$$

If v were constant on the right hand side, then one would expect v to "look locally like" $2\lambda v \Phi$, where

$$\partial_t \Phi = \partial_x^2 \Phi + \partial_x^2 Y^\bullet\ .$$

Idea: Set up fixed point argument in space of functions that "look like Φ" and use the fact that one can define $\Phi\, \partial_x Y^\bullet$ "by hand".

Resulting space is a non-linear algebraic variety embedded in a larger Banach space. Uses controlled rough paths à la Gubinelli-Lyons.

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Conclusions

Take-away message: Nonlinear spaces are required to solve some rough equations pathwise.

Some open problems:

- Extension to $x \in \mathbf{R}$?
- Convergence of microscopic models (for example lattice KPZ) to KPZ. See work with J. Maas and H. Weber.
- Extension to other equations in similar class.
- Rough equations in higher dimensions?

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