Classification and rigidity for von Neumann algebras

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von Neumann algebras (Murray and von Neumann, 1936)

A **von Neumann algebra** is an algebra of bounded operators on a Hilbert space H which is closed under adjoint and in the weak operator topology:

$$T_i \to T \ \text{ w.o.t. } \quad \text{if} \quad \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle, \quad \text{for all} \quad \xi, \eta \in H.$$

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Examples

- $\mathbb{B}(H)$, the algebra of all bounded operators on H.
- $L^\infty(X)$, where (X, μ) is a measure space.
- The commutant of any set of operators that is closed under adjoint.

General constructions of von Neumann algebras

- Γ countable group $\rightsquigarrow$ group von Neumann algebra $L(\Gamma)$.
  Generated by the **left regular representation** of Γ.
  More precisely, $L(\Gamma)$ is the closure of the span of $\{u_g\}_{g\in\Gamma}$,
  where u_g is the unitary operator on $\ell^2(\Gamma)$ given by $u_g(\delta_h) = \delta_{gh}$.

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  where u_g is the unitary operator on $\ell^2(\Gamma)$ given by $u_g(\delta_h) = \delta_{gh}$.
- $\Gamma \curvearrowright (X,\mu)$ measure preserving action of a countable group on a probability space $\rightsquigarrow$ crossed product algebra $L^\infty(X) \rtimes \Gamma$.
  Generated by $L^\infty(X)$ and a copy $\{u_g\}_{g\in\Gamma}$ of the group Γ subject to the relations $u_g a u_g^* = a \circ g^{-1}$.

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Remark

These algebras admit a trace: linear functional satisfying $\tau(ab) = \tau(ba)$.

Classification of II_1 factors

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Proposition

- $L(\Gamma)$ is a II_1 factor if and only if Γ has infinite conjugacy classes (icc).
- $L^\infty(X) \rtimes \Gamma$ is a II_1 factor if $\Gamma \curvearrowright (X, \mu)$ is free and ergodic.

Examples of free ergodic actions

- Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$.
- profinite actions $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$,
  where Γ is a residually finite group and $(\Gamma_n)_n$ is a descending chain of finite index normal subgroups of Γ with $\cap\Gamma_n = \{e\}$.
- the usual actions $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Properties of groups

Amenability

- A group Γ is **amenable** if the left regular representation of Γ has almost invariant vectors. This means that there exist unit vectors $\xi_n \in \ell^2(\Gamma)$ satisfying $\|u_g(\xi_n) - \xi_n\| \to 0$, for all $g \in \Gamma$.
  Recall: $u_g(\delta_h) = \delta_{gh}$.
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Remark. amenable + property (T) $\implies$ finite.

Amenable versus non-amenable

Theorem (Connes, 1975)

- *All* $L(\Gamma)$ *with* Γ *amenable and icc are isomorphic.*
- *All* $L^\infty(X) \rtimes \Gamma$ *with* Γ *infinite amenable and* $\Gamma \curvearrowright (X, \mu)$ *free ergodic are isomorphic.*

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Surprising lack of rigidity: any algebraic property of an amenable group (e.g. being torsion free) and any dynamical property of its actions (e.g. being mixing) is lost when passing to von Neumann algebras.

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Connes (1980) If Γ is an icc property (T) group, then any automorphism of $L(\Gamma)$ that is close to the identity is inner.

Popa's deformation/rigidity theory (2001-)

Basic idea

Study II_1 factors M that have

- a **deformation property**, e.g. M has a 1-parameter group of automorphisms with "good" properties, and
- a **rigidity property**, e.g. M contains $L(\Gamma)$, for a property (T) group Γ.

The combination of these two properties can be sometimes used to understand the structure of M.

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$\rightsquigarrow$ Rigidity results for **crossed product II_1 factors**: when part or even the whole action $\Gamma \curvearrowright (X,\mu)$ can be reconstructed from $L^\infty(X) \rtimes \Gamma$.

Rigidity for crossed product II_1 factors

Definition

Two actions $\Gamma \curvearrowright (X, \mu)$ *are* $\Lambda \curvearrowright (Y, \nu)$ *are called* **conjugate** *if there exist isomorphisms* $\alpha : X \to Y$ *and* $\delta : \Gamma \to \Lambda$ *such that*

$$\alpha(g \cdot x) = \delta(g) \cdot \alpha(x).$$

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- Conjugacy of actions $\Rightarrow$ isomorphism of their von Neumann algebras.
- Rigidity: prove the converse.

Popa's strong rigidity result

Theorem (Popa, 2004)

Let Γ be a property (T) group and $\Gamma \curvearrowright (X, \mu)$ a free ergodic action.
Let Λ be an icc group and $\Lambda \curvearrowright (Y, \nu) = (Y_0, \nu_0)^{\Lambda}$ a Bernoulli action.

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If $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$, then the groups Γ and Λ are isomorphic and their actions are conjugate.

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If $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$, then the groups Γ and Λ are isomorphic and their actions are conjugate.

First result deducing conjugacy of actions from isomorphism of their von Neumann algebras.

Question: can we put all the conditions on one of the actions and allow the other action to be arbitrary?

Superrigidity for crossed product II_1 factors

Definition

A free ergodic action $\Gamma \curvearrowright (X, \mu)$ is called **W*-superrigid** *(or just* **superrigid***) if any action $\Lambda \curvearrowright (Y, \nu)$ such that $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ is conjugate to $\Gamma \curvearrowright (X, \mu)$.*

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Definition

$A \subset M$ **Cartan subalgebra***: maximal abelian von Neumann subalgebra such that the unitary operators* $u \in M$ *satisfying* $uAu^* = A$ *generate* M*.*

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Example*: $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$, for any free ergodic action $\Gamma \curvearrowright (X, \mu)$.*

Remark*. Not every Cartan subalgebras arises this way.*

How to prove that an action $\Gamma \curvearrowright (X, \mu)$ is superrigid

Problem 1

Prove that $L^\infty(X) \rtimes \Gamma$ has a unique (crossed product) Cartan subalgebra, up to unitary conjugacy. In other words, if $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$, then $L^\infty(X)$ and $L^\infty(Y)$ are unitarily conjugated.

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Remark (Singer, 1955): this implies that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are orbit equivalent: there is an isomorphism $\alpha : X \to Y$ satisfying $\alpha(\Gamma \cdot x) = \Lambda \cdot \alpha(x)$.

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Prove that any action that is orbit equivalent to $\Gamma \curvearrowright (X, \mu)$ must be conjugate to it.

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Examples of OE superrigid actions:

- **Furman (1998)**: the actions $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$, for $n \geqslant 3$.
- **Popa (2004)**: Bernoulli actions of property (T) groups.
- more examples: **Monod-Shalom, Kida, Ioana**.

Uniqueness of Cartan subalgebras

Theorem (Ozawa-Popa, 2007)

Let $\mathbb{F}_n \curvearrowright (X, \mu)$ be a free ergodic **profinite** *action of a free group. Then $L^\infty(X) \rtimes \mathbb{F}_n$ has a unique Cartan subalgebra, up to unitary conjugacy.*

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Theorem (Popa-Vaes, 2011-2012)

Let Γ be a free group or any non-elementary hyperbolic group.
Let $\Gamma \curvearrowright (X, \mu)$ be **any** *free ergodic action.*
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Question: what other groups Γ have this property?

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Ioana (2012): any free product $\Gamma = \Gamma_1 * \Gamma_2$, with $|\Gamma_1| \geqslant 2$ and $|\Gamma_2| \geqslant 3$.

The first ℓ^2-Betti number and Cartan subalgebras

Conjecture

Assume that $\beta_1^{(2)}(\Gamma) > 0$.
Then $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra, up to unitary conjugacy, for any free ergodic action $\Gamma \curvearrowright (X, \mu)$.

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Examples of groups with $\beta_1^{(2)} > 0$: free groups, free product groups, groups with n generators and at most $n-2$ relations.

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Partial results: **Popa-Vaes, Chifan-Peterson, Ioana**.

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Remark

If the conjecture holds, then $\beta_1^{(2)}(\Gamma)$ is an invariant of $L^\infty(X) \rtimes \Gamma$.
More precisely, $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda \implies \beta_1^{(2)}(\Gamma) = \beta_1^{(2)}(\Lambda)$.

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Gaboriau (2001): $\beta_1^{(2)}(\Gamma)$ is an orbit equivalence invariant of $\Gamma \curvearrowright (X, \mu)$.

First examples of superrigid actions

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An action $\Gamma \curvearrowright (X,\mu)$ *is called* **superrigid** *if any action* $\Lambda \curvearrowright (Y,\nu)$ *such that* $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ *is conjugate to* $\Gamma \curvearrowright (X,\mu)$*.*

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Theorem (Popa-Vaes, 2009)

First concrete families of superrigid actions:

1. *Bernoulli actions of many amalgamated free product groups.*
2. *any mixing action of $SL_3(\mathbb{Z}) *_{T_3} SL_3(\mathbb{Z})$.*

Superrigidity for Bernoulli actions of property (T) groups

Theorem (Ioana, 2010)

If Γ is any icc property (T) group, then the Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ is superrigid.

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Therefore, for a large of group actions, the crossed product von Neumann algebra $L^\infty(X) \rtimes \Gamma$ remembers the group and the action.

Groups covered by the theorem:

- $\mathrm{PSL}_n(\mathbb{Z})$, for $n \geqslant 3$.
- icc groups with infinite normal subgroups with relative property (T):
  - $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.
  - $\mathrm{SL}_3(\mathbb{Z}) \times \Sigma$, where Σ is any icc group.

Rigidity for group II_1 factors

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Open problems

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Connes' rigidity conjecture $\iff$ icc property (T) groups are superrigid.

First examples of superrigid groups

Theorem (Ioana-Popa-Vaes, 2010)

Define $\Gamma_0 = \mathbb{Z} \ltimes (\bigoplus_{n\in\mathbb{Z}} \mathbb{F}_2)$.
Consider the action $\Gamma_0 \curvearrowright I = \Gamma_0/\mathbb{Z}$ *by left multiplication.*
Then $\Gamma = \Gamma_0 \ltimes (\bigoplus_{i\in I} \mathbb{Z}/2\mathbb{Z})$ *is* **superrigid**.

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More generally, this result holds if we replace $\mathbb{Z}$ by any infinite amenable group, and $\mathbb{F}_2$ by any non-amenable group.