Representations of affine Kac-Moody groups over local and global fields: a survey of some recent results

Alexander Braverman and David Kazhdan

June 29, 2012

p-adic groups and automorphic forms

In this talk we are going to work with a split reductive algebraic group G (e.g. $GL(n)$, $SL(n)$, $SO(n)$, ...). The are many contexts in which one can study representation theory of G.

The context for this talk is as follows: let $\mathcal{K}$ be a local non-archimedian field (i.e. either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_q((s))$ where s is a finite field) with ring of integers $\mathcal{O}$ (e.g. $\mathbb{Z}_p$ or $\mathbb{F}_q[[s]]$). Then the group $G(\mathcal{K})$ is a locally compact topological group with interesting representation theory.

The study of representations of $G(\mathcal{K})$ is important for many reasons; one of the most important being the theory of automorphic forms. These are some representations of $G(\mathbb{A}_F)$ where F is a global field (e.g. a number field) and $\mathbb{A}_F$ is its ring of adeles.

p-adic groups and automorphic forms

In this talk we are going to work with a split reductive algebraic group G (e.g. $GL(n)$, $SL(n)$, $SO(n)$, ...). The are many contexts in which one can study representation theory of G.

The context for this talk is as follows: let $\mathcal{K}$ be a local non-archimedian field (i.e. either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_q((s))$ where s is a finite field) with ring of integers $\mathcal{O}$ (e.g. $\mathbb{Z}_p$ or $\mathbb{F}_q[[s]]$). Then the group $G(\mathcal{K})$ is a locally compact topological group with interesting representation theory.

The study of representations of $G(\mathcal{K})$ is important for many reasons; one of the most important being the theory of automorphic forms. These are some representations of $G(\mathbb{A}_F)$ where F is a global field (e.g. a number field) and $\mathbb{A}_F$ is its ring of adeles.

p-adic groups and automorphic forms

In this talk we are going to work with a split reductive algebraic group G (e.g. $GL(n)$, $SL(n)$, $SO(n)$, ...). The are many contexts in which one can study representation theory of G.
The context for this talk is as follows: let $\mathcal{K}$ be a local non-archimedian field (i.e. either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_q((s))$ where s is a finite field) with ring of integers $\mathcal{O}$ (e.g. $\mathbb{Z}_p$ or $\mathbb{F}_q[[s]]$).
Then the group $G(\mathcal{K})$ is a locally compact topological group with interesting representation theory.
The study of representations of $G(\mathcal{K})$ is important for many reasons; one of the most important being the theory of automorphic forms.
These are some representations of $G(\mathbb{A}_F)$ where F is a global field (e.g. a number field) and $\mathbb{A}_F$ is its ring of adeles.

Langlands correspondence

In both (local and global) cases the most interesting statement about the above representation is *Langlands correspondence* which relates representations of $G(\mathcal{K})$ (resp. automorphic representations of $G(\mathbb{A}_F)$) to homomorphisms from the absolute Galois group of $\mathcal{K}$ (resp. of F) to the Langlads dual group $G^\vee$ of G.

Our dream is to generalize this to affine Kac-Moody groups. What are they?

Langlands correspondence

In both (local and global) cases the most interesting statement about the above representation is *Langlands correspondence* which relates representations of $G(\mathcal{K})$ (resp. automorphic representations of $G(\mathbb{A}_F)$) to homomorphisms from the absolute Galois group of $\mathcal{K}$ (resp. of F) to the Langlads dual group $G^\vee$ of G.
Our dream is to generalize this to affine Kac-Moody groups. What are they?

Affine Kac-Moody groups

Consider the polynomial loop group $G[t, t^{-1}]$ (this is an infinite-dimensional group ind-scheme).

One can define a central extension $\widetilde{G}$ of $G[t, t^{-1}]$:

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G[t, t^{-1}] \to 1$$

The multiplicative group $\mathbb{G}_m$ acts naturally on $G[t, t^{-1}]$ and $\widetilde{G}$ "by loop rotation". Set

$$G_{\text{aff}} = \widetilde{G} \rtimes \mathbb{G}_m, ; \mathfrak{g}_{\text{aff}} = \text{Lie}(G_{\text{aff}}).$$

Similarly, one can consider the corresponding completed affine Kac-Moody group $\widehat{G}_{\text{aff}}$ by replacing the polynomial loop group $G[t, t^{-1}]$ with the formal loop group $G((t))$.

Affine Kac-Moody groups

Consider the polynomial loop group $G[t, t^{-1}]$ (this is an infinite-dimensional group ind-scheme).
One can define a central extension $\widetilde{G}$ of $G[t, t^{-1}]$:

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G[t, t^{-1}] \to 1$$

The multiplicative group $\mathbb{G}_m$ acts naturally on $G[t, t^{-1}]$ and $\widetilde{G}$ "by loop rotation". Set

$$G_{\text{aff}} = \widetilde{G} \rtimes \mathbb{G}_m, ; \mathfrak{g}_{\text{aff}} = \text{Lie}(G_{\text{aff}}).$$

Similarly, one can consider the corresponding completed affine Kac-Moody group $\widehat{G}_{\text{aff}}$ by replacing the polynomial loop group $G[t, t^{-1}]$ with the formal loop group $G((t))$.

Affine Kac-Moody groups

Consider the polynomial loop group $G[t, t^{-1}]$ (this is an infinite-dimensional group ind-scheme).
One can define a central extension $\widetilde{G}$ of $G[t, t^{-1}]$:

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G[t, t^{-1}] \to 1$$

The multiplicative group $\mathbb{G}_m$ acts naturally on $G[t, t^{-1}]$ and $\widetilde{G}$ "by loop rotation". Set

$$G_{\mathrm{aff}} = \widetilde{G} \rtimes \mathbb{G}_m, ; \mathfrak{g}_{\mathrm{aff}} = \mathrm{Lie}(G_{\mathrm{aff}}).$$

Similarly, one can consider the corresponding completed affine Kac-Moody group $\widehat{G}_{\mathrm{aff}}$ by replacing the polynomial loop group $G[t, t^{-1}]$ with the formal loop group $G((t))$.

Affine Kac-Moody groups

Consider the polynomial loop group $G[t, t^{-1}]$ (this is an infinite-dimensional group ind-scheme).
One can define a central extension $\widetilde{G}$ of $G[t, t^{-1}]$:

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G[t, t^{-1}] \to 1$$

The multiplicative group $\mathbb{G}_m$ acts naturally on $G[t, t^{-1}]$ and $\widetilde{G}$ "by loop rotation". Set

$$G_{\text{aff}} = \widetilde{G} \rtimes \mathbb{G}_m, ; \mathfrak{g}_{\text{aff}} = \text{Lie}(G_{\text{aff}}).$$

Similarly, one can consider the corresponding completed affine Kac-Moody group $\widehat{G}_{\text{aff}}$ by replacing the polynomial loop group $G[t, t^{-1}]$ with the formal loop group $G((t))$.

The dream

As was mentioned above, our dream would be to develop an analog of the above representation theories and the Langlands correspondence for the group G_{aff} or $\widehat{G}_{\mathrm{aff}}$ (or, more generally, for any symmetrizable Kac-Moody group). This is a fascinating task by itself but we also believe that a fully developed theory of automorphic forms for $\widehat{G}_{\mathrm{aff}}$ will have powerful applications to automorphic forms on G (for example using the theory of Eisenstein series).

This talk

At the moment the above dream remains only a dream; however, in the recent years some interesting results about representation theory of G_{aff} over either local or global field have appeared. The purpose of this talk is to survey some of those results; more precisely, we are going to concentrate on two aspects: study of some particular Hecke algebras in the local case and the study of Eisenstein series in the global case.

All the results that we are going to discuss generalize well-known results for the group G itself; however, the generalizations are not always straightforward and some new features appear in the affine case.

Specifically, we are going to discuss the following subjects: Hecke algebras for G_{aff}, the affine Satake isomorphism and affine Eisenstein series.

This talk

At the moment the above dream remains only a dream; however, in the recent years some interesting results about representation theory of G_{aff} over either local or global field have appeared. The purpose of this talk is to survey some of those results; more precisely, we are going to concentrate on two aspects: study of some particular Hecke algebras in the local case and the study of Eisenstein series in the global case. All the results that we are going to discuss generalize well-known results for the group G itself; however, the generalizations are not always straightforward and some new features appear in the affine case.

Specifically, we are going to discuss the following subjects: Hecke algebras for G_{aff}, the affine Satake isomorphism and affine Eisenstein series.

This talk

At the moment the above dream remains only a dream; however, in the recent years some interesting results about representation theory of G_{aff} over either local or global field have appeared. The purpose of this talk is to survey some of those results; more precisely, we are going to concentrate on two aspects: study of some particular Hecke algebras in the local case and the study of Eisenstein series in the global case. All the results that we are going to discuss generalize well-known results for the group G itself; however, the generalizations are not always straightforward and some new features appear in the affine case.
Specifically, we are going to discuss the following subjects: Hecke algebras for G_{aff}, the affine Satake isomorphism and affine Eisenstein series.

Hecke algebras

For any open compact subgroup K of $G(\mathcal{K})$, we denote by $\mathcal{H}(G, K)$ the Hecke algebra of G with respect to K. A choice of a Haar measure on $G(\mathcal{K})$ provides an identification of $\mathcal{H}(G, K)$ with the space of K-bi-invariant functions on $G(\mathcal{K})$.

For every K as above, there is a natural functor from $\mathcal{M}(G(\mathcal{K}))$ to the category of left $\mathcal{H}(G, K)$-modules, sending every representation V to the corresponding space V^K of K-invariants and one can understand the category $\mathcal{M}(G, K)$ by studying $\mathcal{H}(G, K)$-modules for different K. The example of K which will be of special interest to us is $K = G(\mathcal{O})$ (which is a maximal compact subgroup of $G(\mathcal{K})$). The corresponding Hecke algebra in this case is called *the spherical Hecke algebra* and it is denoted it by $\mathcal{H}_{\mathrm{sph}}(G, \mathcal{K})$. Other cases will be discussed later.

Hecke algebras

For any open compact subgroup K of $G(\mathcal{K})$, we denote by $\mathcal{H}(G,K)$ the Hecke algebra of G with respect to K. A choice of a Haar measure on $G(\mathcal{K})$ provides an identification of $\mathcal{H}(G,K)$ with the space of K-bi-invariant functions on $G(\mathcal{K})$.
For every K as above, there is a natural functor from $\mathcal{M}(G(\mathcal{K}))$ to the category of left $\mathcal{H}(G,K)$-modules, sending every representation V to the corresponding space V^K of K-invariants and one can understand the category $\mathcal{M}(G,K)$ by studying $\mathcal{H}(G,K)$-modules for different K.
The example of K which will be of special interest to us is $K = G(\mathcal{O})$ (which is a maximal compact subgroup of $G(\mathcal{K})$). The corresponding Hecke algebra in this case is called *the spherical Hecke algebra* and it is denoted it by $\mathcal{H}_{\mathrm{sph}}(G,\mathcal{K})$. Other cases will be discussed later.

For any open compact subgroup K of $G(\mathcal{K})$, we denote by $\mathcal{H}(G,K)$ the Hecke algebra of G with respect to K. A choice of a Haar measure on $G(\mathcal{K})$ provides an identification of $\mathcal{H}(G,K)$ with the space of K-bi-invariant functions on $G(\mathcal{K})$.
For every K as above, there is a natural functor from $\mathcal{M}(G(\mathcal{K}))$ to the category of left $\mathcal{H}(G,K)$-modules, sending every representation V to the corresponding space V^K of K-invariants and one can understand the category $\mathcal{M}(G,K)$ by studying $\mathcal{H}(G,K)$-modules for different K.
The example of K which will be of special interest to us is $K = G(\mathcal{O})$ (which is a maximal compact subgroup of $G(\mathcal{K})$). The corresponding Hecke algebra in this case is called *the spherical Hecke algebra* and it is denoted it by $\mathcal{H}_{\mathrm{sph}}(G,\mathcal{K})$. Other cases will be discussed later.

Satake isomorphism

The spherical Hecke algebra $\mathcal{H}_{\text{sph}}(G, \mathcal{K})$ is commutative. To describe it let us denote by T a maximal torus of G and by $T^\vee$ the dual torus over $\mathbb{C}$. We shall also denote by Λ the weight lattice of $T^\vee$ and by W the Weyl group.

Example. When $G = GL(n)$ we have $T^\vee = (\mathbb{C}^*)^n$, $\Lambda = \mathbb{Z}^n$ and $W = S_n$.

Theorem (Satake isomorphism)

There is a canonical isomorphism between $\mathcal{H}_{\text{sph}}(G, \mathcal{K})$ and the algebra $\mathbb{C}(T^\vee)^W$ of W-invariant polynomial functions on $T^\vee$.

Here "canonical" means that there is an explicit construction of the isomorphism.

Satake isomorphism

The spherical Hecke algebra $\mathcal{H}_{\text{sph}}(G,\mathcal{K})$ is commutative. To describe it let us denote by T a maximal torus of G and by $T^\vee$ the dual torus over $\mathbb{C}$. We shall also denote by Λ the weight lattice of $T^\vee$ and by W the Weyl group.

Example. When $G = GL(n)$ we have $T^\vee = (\mathbb{C}^*)^n$, $\Lambda = \mathbb{Z}^n$ and $W = S_n$.

Theorem (Satake isomorphism)

There is a canonical isomorphism between $\mathcal{H}_{\text{sph}}(G,\mathcal{K})$ and the algebra $\mathbb{C}(T^\vee)^W$ of W-invariant polynomial functions on $T^\vee$.

Here "canonical" means that there is an explicit construction of the isomorphism.

Satake isomorphism

The spherical Hecke algebra $\mathcal{H}_{\mathrm{sph}}(G,\mathcal{K})$ is commutative. To describe it let us denote by T a maximal torus of G and by $T^\vee$ the dual torus over $\mathbb{C}$. We shall also denote by Λ the weight lattice of $T^\vee$ and by W the Weyl group.

Example. When $G = GL(n)$ we have $T^\vee = (\mathbb{C}^*)^n$, $\Lambda = \mathbb{Z}^n$ and $W = S_n$.

Theorem (Satake isomorphism)

There is a canonical isomorphism between $\mathcal{H}_{\mathrm{sph}}(G,\mathcal{K})$ and the algebra $\mathbb{C}(T^\vee)^W$ of W-invariant polynomial functions on $T^\vee$.

Here "canonical" means that there is an explicit construction of the isomorphism.

Interpretation via $G^\vee$

The algebra $\mathbb{C}(T^\vee)^W$ is also naturally isomorphic to the complexified Grothendieck ring $K_0(\text{Rep}(G^\vee))$ of finite-dimensional representations of the Langlands dual group $G^\vee$ (the corresponding map from $K_0(\text{Rep}(G^\vee))$ to $\mathbb{C}(T^\vee)^W$ is the character map).
Thus the Satake isomorphism can be viewed as an isomorphism

$$\mathcal{H}_{\text{sph}}(G, \mathcal{K}) \simeq K_0(\text{Rep}(G^\vee)).$$

This interpretation is important for Langlands correspondence; in fact, the Satake isomorphism is one of the starting points for the Langlands conjectures.
We are going to present a generalization of the Satake isomorphism to the case of affine Kac-Moody groups later in the talk.

Interpretation via $G^\vee$

The algebra $\mathbb{C}(T^\vee)^W$ is also naturally isomorphic to the complexified Grothendieck ring $K_0(\mathrm{Rep}(G^\vee))$ of finite-dimensional representations of the Langlands dual group $G^\vee$ (the corresponding map from $K_0(\mathrm{Rep}(G^\vee))$ to $\mathbb{C}(T^\vee)^W$ is the character map).
Thus the Satake isomorphism can be viewed as an isomorphism

$$\mathcal{H}_{\mathrm{sph}}(G, \mathcal{K}) \simeq K_0(\mathrm{Rep}(G^\vee)).$$

This interpretation is important for Langlands correspondence; in fact, the Satake isomorphism is one of the starting points for the Langlands conjectures.
We are going to present a generalization of the Satake isomorphism to the case of affine Kac-Moody groups later in the talk.

Interpretation via $G^\vee$

The algebra $\mathbb{C}(T^\vee)^W$ is also naturally isomorphic to the complexified Grothendieck ring $K_0(\mathrm{Rep}(G^\vee))$ of finite-dimensional representations of the Langlands dual group $G^\vee$ (the corresponding map from $K_0(\mathrm{Rep}(G^\vee))$ to $\mathbb{C}(T^\vee)^W$ is the character map).
Thus the Satake isomorphism can be viewed as an isomorphism

$$\mathcal{H}_{\mathrm{sph}}(G,\mathcal{K}) \simeq K_0(\mathrm{Rep}(G^\vee)).$$

This interpretation is important for Langlands correspondence; in fact, the Satake isomorphism is one of the starting points for the Langlands conjectures.
We are going to present a generalization of the Satake isomorphism to the case of affine Kac-Moody groups later in the talk.

Macdonald formula

The algebra $\mathcal{H}_{\mathrm{sph}}(G, \mathcal{K})$ has a natural basis.
Namely, let $\varpi \in \mathcal{O}$ be a uniformizer; for $\lambda \in \Lambda$ let us denote by ϖ^λ the image of $\varpi \in \mathcal{K}^*$ under the map $\lambda : \mathcal{K}^* \to T(\mathcal{K}) \subset G(\mathcal{K})$.
Then $G(\mathcal{K})$ is the disjoint union of the cosets $G(\mathcal{O}) \cdot \varpi^\lambda \cdot G(\mathcal{O})$ when λ runs over Λ_+. For every $\lambda \in \Lambda_+$ we denote by $h_\lambda \in \mathcal{H}_{\mathrm{sph}}(G, \mathcal{K})$ the characteristic function of the corresponding double coset. This is a basis of $\mathcal{H}_{\mathrm{sph}}(G, \mathcal{K})$.
Question: What happens to this basis under Satake isomorphism?
Answer was given by Macdonald in 1968.
Let W_λ is the stabilizer of λ in W and set

$$W_\lambda(q) = \sum_{w \in W_\lambda} q^{\ell(w)}.$$

Macdonald formula

The algebra $\mathcal{H}_{\text{sph}}(G, \mathcal{K})$ has a natural basis.
Namely, let $\varpi \in \mathcal{O}$ be a uniformizer; for $\lambda \in \Lambda$ let us denote by ϖ^λ the image of $\varpi \in \mathcal{K}^*$ under the map $\lambda : \mathcal{K}^* \to T(\mathcal{K}) \subset G(\mathcal{K})$.
Then $G(\mathcal{K})$ is the disjoint union of the cosets $G(\mathcal{O}) \cdot \varpi^\lambda \cdot G(\mathcal{O})$ when λ runs over Λ_+. For every $\lambda \in \Lambda_+$ we denote by $h_\lambda \in \mathcal{H}_{\text{sph}}(G, \mathcal{K})$ the characteristic function of the corresponding double coset. This is a basis of $\mathcal{H}_{\text{sph}}(G, \mathcal{K})$.
Question: What happens to this basis under Satake isomorphism?
Answer was given by Macdonald in 1968.
Let W_λ is the stabilizer of λ in W and set

$$W_\lambda(q) = \sum_{w \in W_\lambda} q^{\ell(w)}.$$

Macdonald formula

The algebra $\mathcal{H}_{\text{sph}}(G, \mathcal{K})$ has a natural basis.
Namely, let $\varpi \in \mathcal{O}$ be a uniformizer; for $\lambda \in \Lambda$ let us denote by ϖ^λ the image of $\varpi \in \mathcal{K}^*$ under the map $\lambda : \mathcal{K}^* \to T(\mathcal{K}) \subset G(\mathcal{K})$.
Then $G(\mathcal{K})$ is the disjoint union of the cosets $G(\mathcal{O}) \cdot \varpi^\lambda \cdot G(\mathcal{O})$ when λ runs over Λ_+. For every $\lambda \in \Lambda_+$ we denote by $h_\lambda \in \mathcal{H}_{\text{sph}}(G, \mathcal{K})$ the characteristic function of the corresponding double coset. This is a basis of $\mathcal{H}_{\text{sph}}(G, \mathcal{K})$.
Question: What happens to this basis under Satake isomorphism?
Answer was given by Macdonald in 1968.
Let W_λ is the stabilizer of λ in W and set

$$W_\lambda(q) = \sum_{w \in W_\lambda} q^{\ell(w)}.$$

Macdonald formula

The algebra $\mathcal{H}_{\mathrm{sph}}(G,\mathcal{K})$ has a natural basis.
Namely, let $\varpi \in \mathcal{O}$ be a uniformizer; for $\lambda \in \Lambda$ let us denote by ϖ^λ the image of $\varpi \in \mathcal{K}^*$ under the map $\lambda : \mathcal{K}^* \to T(\mathcal{K}) \subset G(\mathcal{K})$.
Then $G(\mathcal{K})$ is the disjoint union of the cosets $G(\mathcal{O}) \cdot \varpi^\lambda \cdot G(\mathcal{O})$ when λ runs over Λ_+. For every $\lambda \in \Lambda_+$ we denote by $h_\lambda \in \mathcal{H}_{\mathrm{sph}}(G,\mathcal{K})$ the characteristic function of the corresponding double coset. This is a basis of $\mathcal{H}_{\mathrm{sph}}(G,\mathcal{K})$.
Question: What happens to this basis under Satake isomorphism?
Answer was given by Macdonald in 1968.
Let W_λ is the stabilizer of λ in W and set

$$W_\lambda(q) = \sum_{w \in W_\lambda} q^{\ell(w)}.$$

Theorem

$$\mathcal{S}(h_\lambda) = \frac{q^{\langle \lambda, \rho^\vee \rangle}}{W_\lambda(q^{-1})} \sum_{w \in W} w\left(e^\lambda \frac{\prod\limits_{\alpha \in R_+} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_+} 1 - e^{-\alpha}} \right).$$

Here $\rho^\vee$ is the half-sum of the positive roots of G.

Reformulation: For $\lambda \in \Lambda_+$ let $L(\lambda)$ denote the irreducible representation of $G^\vee$ with highest weight λ.

Theorem

Let $\lambda, \mu \in \Lambda_+$. Then $\mathcal{S}^{-1}([L(\lambda)])(\varpi^\mu)$ is a certain q-analog of the weight multiplicity $\dim L(\lambda)_\mu$ *(i.e. it is a polynomial in q with integral coefficients whose value at $q = 1$ is equal to the weight multiplicity).*

Theorem

$$\mathcal{S}(h_\lambda) = \frac{q^{\langle \lambda, \rho^\vee \rangle}}{W_\lambda(q^{-1})} \sum_{w \in W} w \left(e^\lambda \frac{\prod\limits_{\alpha \in R_+} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_+} 1 - e^{-\alpha}} \right).$$

Here $\rho^\vee$ is the half-sum of the positive roots of G.

Reformulation: For $\lambda \in \Lambda_+$ let $L(\lambda)$ denote the irreducible representation of $G^\vee$ with highest weight λ.

Theorem

Let $\lambda, \mu \in \Lambda_+$. Then $\mathcal{S}^{-1}([L(\lambda)])(\varpi^\mu)$ is a certain q-analog of the weight multiplicity $\dim L(\lambda)_\mu$ *(i.e. it is a polynomial in q with integral coefficients whose value at $q = 1$ is equal to the weight multiplicity).*

Theorem

$$\mathcal{S}(h_\lambda) = \frac{q^{\langle\lambda,\rho^\vee\rangle}}{W_\lambda(q^{-1})} \sum_{w\in W} w\left(e^\lambda \frac{\prod\limits_{\alpha\in R_+} 1 - q^{-1}e^{-\alpha}}{\prod\limits_{\alpha\in R_+} 1 - e^{-\alpha}} \right).$$

Here $\rho^\vee$ is the half-sum of the positive roots of G.

Reformulation: For $\lambda \in \Lambda_+$ let $L(\lambda)$ denote the irreducible representation of $G^\vee$ with highest weight λ.

Theorem

Let $\lambda, \mu \in \Lambda_+$. Then $\mathcal{S}^{-1}([L(\lambda)])(\varpi^\mu)$ is a certain q-analog of the weight multiplicity $\dim L(\lambda)_\mu$ *(i.e. it is a polynomial in q with integral coefficients whose value at $q = 1$ is equal to the weight multiplicity).*

Macdonald's formula is used in the work of Langlands on Eisenstein series in the theory of automorphic forms; for example a special case of it gives the so called *Gindikin-Karpelevich formula*, which is responsible for Langlands' computations of the constant term of Eisenstein series.

Let us now turn to the affine case.

Macdonald's formula is used in the work of Langlands on Eisenstein series in the theory of automorphic forms; for example a special case of it gives the so called *Gindikin-Karpelevich formula*, which is responsible for Langlands' computations of the constant term of Eisenstein series.

Let us now turn to the affine case.

The spherical Hecke algebra for G_{aff}

Want: To define $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$.
Note that the groups $G_{\text{aff}}(\mathcal{O})$ and $G_{\text{aff}}(\mathcal{K})$ make sense.
Bad news: The convolution of $G_{\text{aff}}(\mathcal{O})$ double cosets does not make sense - things are infinite...
Good news: Sometimes the convolution does make sense!
The group G_{aff} by definition maps to $\mathbb{G}_m$; thus $G_{\text{aff}}(\mathcal{K})$ maps to $\mathcal{K}^*$. We denote by ϱ the composition of this map with the valuation map $\mathcal{K}^* \to \mathbb{Z}$.
Let $G^+_{\text{aff}}(\mathcal{K})$ be the subsemigroup of $G_{\text{aff}}(\mathcal{K})$ denerated by:
- the central $\mathcal{K}^* \subset G_{\text{aff}}(\mathcal{K})$;
- the subgroup $G_{\text{aff}}(\mathcal{O})$;
- All elements $g \in G_{\text{aff}}(\mathcal{K})$ such that $\varrho(g) > 0$.

The spherical Hecke algebra for G_{aff}

Want: To define $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$.
Note that the groups $G_{\mathrm{aff}}(\mathcal{O})$ and $G_{\mathrm{aff}}(\mathcal{K})$ make sense.
Bad news: The convolution of $G_{\mathrm{aff}}(\mathcal{O})$ double cosets does not make sense - things are infinite...
Good news: Sometimes the convolution does make sense!
The group G_{aff} by definition maps to $\mathbb{G}_m$; thus $G_{\mathrm{aff}}(\mathcal{K})$ maps to $\mathcal{K}^*$. We denote by ϱ the composition of this map with the valuation map $\mathcal{K}^* \to \mathbb{Z}$.
Let $G^+_{\mathrm{aff}}(\mathcal{K})$ be the subsemigroup of $G_{\mathrm{aff}}(\mathcal{K})$ denerated by:
- the central $\mathcal{K}^* \subset G_{\mathrm{aff}}(\mathcal{K})$;
- the subgroup $G_{\mathrm{aff}}(\mathcal{O})$;
- All elements $g \in G_{\mathrm{aff}}(\mathcal{K})$ such that $\varrho(g) > 0$.

The spherical Hecke algebra for G_{aff}

Want: To define $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$.
Note that the groups $G_{\text{aff}}(\mathcal{O})$ and $G_{\text{aff}}(\mathcal{K})$ make sense.
Bad news: The convolution of $G_{\text{aff}}(\mathcal{O})$ double cosets does not make sense - things are infinite...
Good news: Sometimes the convolution does make sense!
The group G_{aff} by definition maps to $\mathbb{G}_m$; thus $G_{\text{aff}}(\mathcal{K})$ maps to $\mathcal{K}^*$. We denote by ϱ the composition of this map with the valuation map $\mathcal{K}^* \to \mathbb{Z}$.
Let $G^+_{\text{aff}}(\mathcal{K})$ be the subsemigroup of $G_{\text{aff}}(\mathcal{K})$ denerated by:
• the central $\mathcal{K}^* \subset G_{\text{aff}}(\mathcal{K})$;
• the subgroup $G_{\text{aff}}(\mathcal{O})$;
• All elements $g \in G_{\text{aff}}(\mathcal{K})$ such that $\varrho(g) > 0$.

The spherical Hecke algebra for G_{aff}

Want: To define $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$.
Note that the groups $G_{\text{aff}}(\mathcal{O})$ and $G_{\text{aff}}(\mathcal{K})$ make sense.
Bad news: The convolution of $G_{\text{aff}}(\mathcal{O})$ double cosets does not make sense - things are infinite...
Good news: Sometimes the convolution does make sense!
The group G_{aff} by definition maps to $\mathbb{G}_m$; thus $G_{\text{aff}}(\mathcal{K})$ maps to $\mathcal{K}^*$. We denote by ϱ the composition of this map with the valuation map $\mathcal{K}^* \to \mathbb{Z}$.
Let $G^+_{\text{aff}}(\mathcal{K})$ be the subsemigroup of $G_{\text{aff}}(\mathcal{K})$ denerated by:

- the central $\mathcal{K}^* \subset G_{\text{aff}}(\mathcal{K})$;
- the subgroup $G_{\text{aff}}(\mathcal{O})$;
- All elements $g \in G_{\text{aff}}(\mathcal{K})$ such that $\varrho(g) > 0$.

The spherical Hecke algera for G_{aff}

Theorem (Braverman-Kazhdan)

There exists an associative algebra structure on a completion $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ of the space of finite linear combinations of $G_{\mathrm{aff}}(\mathcal{O})$-double cosets on $G_{\mathrm{aff}}(\mathcal{K})^+$.

This statement is non-trivial – originally it was proved using some cumbersome algebro-geometric machinery.

Two other proofs have appeared since then:

- Gaussent and Rousseau - using an affine version of the building (works for any symmetrizable Kac-Moody group!)
- Braverman, Kazhdan, Patnaik - an elementary (but long) proof

The spherical Hecke algera for G_{aff}

Theorem (Braverman-Kazhdan)

There exists an associative algebra structure on a completion $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$ of the space of finite linear combinations of $G_{\text{aff}}(\mathcal{O})$-double cosets on $G_{\text{aff}}(\mathcal{K})^+$.

This statement is non-trivial – originally it was proved using some cumbersome algebro-geometric machinery.

Two other proofs have appeared since then:

- Gaussent and Rousseau - using an affine version of the building (works for any symmetrizable Kac-Moody group!)
- Braverman, Kazhdan, Patnaik - an elementary (but long) proof

The spherical Hecke algera for G_{aff}

Theorem (Braverman-Kazhdan)

There exists an associative algebra structure on a completion $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ of the space of finite linear combinations of $G_{\mathrm{aff}}(\mathcal{O})$-double cosets on $G_{\mathrm{aff}}(\mathcal{K})^+$.

This statement is non-trivial – originally it was proved using some cumbersome algebro-geometric machinery.

Two other proofs have appeared since then:
• Gaussent and Rousseau - using an affine version of the building (works for any symmetrizable Kac-Moody group!)
• Braverman, Kazhdan, Patnaik - an elementary (but long) proof

Structure of $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$

The algebra $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ is graded by $\mathbb{Z}_{\geq 0}$ (the grading comes from the natural map $\varrho : G_{\mathrm{aff}}(\mathcal{K})^+ \to \mathbb{Z}_{\geq 0}$ which is invariant with respect to (left and right) multiplication by $G_{\mathrm{aff}}(\mathcal{O})$).
It is also an algebra over the field $\mathbb{C}((v))$ of Laurent power series in a variable v, which comes from the central $\mathcal{K}^*$ in $G_{\mathrm{aff}}(\mathcal{K})$.
Example. Let $G = \mathbb{G}_m$. Then $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ is isomorphic (as a graded algebra) to the coordinate ring of the Tate elliptic curve E_v.

What about affine Satake? LHS is already defined - what about RHS?

Structure of $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$

The algebra $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$ is graded by $\mathbb{Z}_{\geq 0}$ (the grading comes from the natural map $\varrho : G_{\text{aff}}(\mathcal{K})^+ \to \mathbb{Z}_{\geq 0}$ which is invariant with respect to (left and right) multiplication by $G_{\text{aff}}(\mathcal{O})$).
It is also an algebra over the field $\mathbb{C}((v))$ of Laurent power series in a variable v, which comes from the central $\mathcal{K}^*$ in $G_{\text{aff}}(\mathcal{K})$.
Example. Let $G = \mathbb{G}_m$. Then $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$ is isomorphic (as a graded algebra) to the coordinate ring of the Tate elliptic curve E_v.

What about affine Satake? LHS is already defined - what about RHS?

Structure of $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$

The algebra $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ is graded by $\mathbb{Z}_{\geq 0}$ (the grading comes from the natural map $\varrho : G_{\mathrm{aff}}(\mathcal{K})^+ \to \mathbb{Z}_{\geq 0}$ which is invariant with respect to (left and right) multiplication by $G_{\mathrm{aff}}(\mathcal{O})$).
It is also an algebra over the field $\mathbb{C}((v))$ of Laurent power series in a variable v, which comes from the central $\mathcal{K}^*$ in $G_{\mathrm{aff}}(\mathcal{K})$.
Example. Let $G = \mathbb{G}_m$. Then $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ is isomorphic (as a graded algebra) to the coordinate ring of the Tate elliptic curve E_v.

What about affine Satake? LHS is already defined - what about RHS?

Satake isomorphism for G_{aff}

Let $T^\vee_{\mathrm{aff}} = \mathbb{G}_m \times T^\vee \times \mathbb{G}_m$ be the dual affine torus. It carries a natural action of $W_{\mathrm{aff}} = W \ltimes \Lambda$. One can define a graded $\mathbb{C}((v))$-algebra $\mathbb{C}(\widehat{T}^\vee_{\mathrm{aff}})^{W_{\mathrm{aff}}}$ - the space of W_{aff}-invariants in a certain completion of $\mathbb{C}(T^\vee_{\mathrm{aff}})$ (no invariants before completion!)

Theorem (Braverman-Kazhdan)

The algebra $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ is canonically isomorphic to $\mathbb{C}(\widehat{T}^\vee_{\mathrm{aff}})^{W_{\mathrm{aff}}}$.

The construction of the isomorphism is similar to the usual case (but it is more tricky to check that everything is well-defined).

Satake isomorphism for G_{aff}

Let $T^\vee_{\text{aff}} = \mathbb{G}_m \times T^\vee \times \mathbb{G}_m$ be the dual affine torus. It carries a natural action of $W_{\text{aff}} = W \ltimes \Lambda$. One can define a graded $\mathbb{C}((v))$-algebra $\mathbb{C}(\widehat{T}^\vee_{\text{aff}})^{W_{\text{aff}}}$ - the space of W_{aff}-invariants in a certain completion of $\mathbb{C}(T^\vee_{\text{aff}})$ (no invariants before completion!)

Theorem (Braverman-Kazhdan)

The algebra $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$ is canonically isomorphic to $\mathbb{C}(\widehat{T}^\vee_{\text{aff}})^{W_{\text{aff}}}$.

The construction of the isomorphism is similar to the usual case (but it is more tricky to check that everything is well-defined).

Satake isomorphism for G_{aff}

Let $T_{\text{aff}}^\vee = \mathbb{G}_m \times T^\vee \times \mathbb{G}_m$ be the dual affine torus. It carries a natural action of $W_{\text{aff}} = W \ltimes \Lambda$. One can define a graded $\mathbb{C}((v))$-algebra $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}$ - the space of W_{aff}-invariants in a certain completion of $\mathbb{C}(T_{\text{aff}}^\vee)$ (no invariants before completion!)

Theorem (Braverman-Kazhdan)

The algebra $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$ is canonically isomorphic to $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}$.

The construction of the isomorphism is similar to the usual case (but it is more tricky to check that everything is well-defined).

Interpretation via Langlands dual group

Assume that G is simple and simply connected. Define the *Langlands dual group* $G^\vee_{\text{aff}}$, which is a group ind-scheme over $\mathbb{C}$. $G^\vee_{\text{aff}}$ is another Kac-Moody group whose Lie algebra $\mathfrak{g}^\vee_{\text{aff}}$ is an affine Kac-Moody algebra with root system dual to that of $\mathfrak{g}_{\text{aff}}$ - can be a twisted affine root system!

$\exists$ Rep$(G^\vee_{\text{aff}})$ - contains all highest weight integrable representations of finite length and also certain infinite direct sums - to make it stable under tensor product! The character map provides an isomorphism of the complexified Grothendieck ring $K_0(G^\vee_{\text{aff}})$ with $\mathbb{C}(\widehat{T}^\vee_{\text{aff}})^{W_{\text{aff}}}$.

The grading on $K_0(G_{\text{aff}})$ comes from the central charge of $G^\vee_{\text{aff}}$-modules and the action of the variable v comes from tensoring $G^\vee_{\text{aff}}$-modules by the one-dimensional representation coming from the homomorphism $G^\vee_{\text{aff}} \to \mathbb{C}^*$.

Interpretation via Langlands dual group

Assume that G is simple and simply connected. Define the *Langlands dual group* $G_{\mathrm{aff}}^{\vee}$, which is a group ind-scheme over $\mathbb{C}$. $G_{\mathrm{aff}}^{\vee}$ is another Kac-Moody group whose Lie algebra $\mathfrak{g}_{\mathrm{aff}}^{\vee}$ is an affine Kac-Moody algebra with root system dual to that of $\mathfrak{g}_{\mathrm{aff}}$ - can be a twisted affine root system!
$\exists$ $\mathrm{Rep}(G_{\mathrm{aff}}^{\vee})$ - contains all highest weight integrable representations of finite length and also certain infinite direct sums - to make it stable under tensor product! The character map provides an isomorphism of the complexified Grothendieck ring $K_0(G_{\mathrm{aff}}^{\vee})$ with $\mathbb{C}(\widehat{T}_{\mathrm{aff}}^{\vee})^{W_{\mathrm{aff}}}$.
The grading on $K_0(G_{\mathrm{aff}})$ comes from the central charge of $G_{\mathrm{aff}}^{\vee}$-modules and the action of the variable v comes from tensoring $G_{\mathrm{aff}}^{\vee}$-modules by the one-dimensional representation coming from the homomorphism $G_{\mathrm{aff}}^{\vee} \to \mathbb{C}^*$.

Interpretation via Langlands dual group

Assume that G is simple and simply connected. Define the *Langlands dual group* $G^\vee_{\text{aff}}$, which is a group ind-scheme over $\mathbb{C}$. $G^\vee_{\text{aff}}$ is another Kac-Moody group whose Lie algebra $\mathfrak{g}^\vee_{\text{aff}}$ is an affine Kac-Moody algebra with root system dual to that of $\mathfrak{g}_{\text{aff}}$ - can be a twisted affine root system!
$\exists$ Rep($G^\vee_{\text{aff}}$) - contains all highest weight integrable representations of finite length and also certain infinite direct sums - to make it stable under tensor product! The character map provides an isomorphism of the complexified Grothendieck ring $K_0(G^\vee_{\text{aff}})$ with $\mathbb{C}(\widehat{T}^\vee_{\text{aff}})^{W_{\text{aff}}}$.
The grading on $K_0(G_{\text{aff}})$ comes from the central charge of $G^\vee_{\text{aff}}$-modules and the action of the variable v comes from tensoring $G^\vee_{\text{aff}}$-modules by the one-dimensional representation coming from the homomorphism $G^\vee_{\text{aff}} \to \mathbb{C}^*$.

Affine Macdonald formula

Want: Analog of Macdonald formula in the affine case.

The (topological) basis $\{h_\lambda\}$ of $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}}, \mathcal{K})$ can be defined as in the finite-dimensional case. One might expect that

$$\mathcal{S}_{\mathrm{aff}}(h_\lambda) = \frac{q^{\langle \lambda, \rho_{\mathrm{aff}}^\vee \rangle}}{W_{\mathrm{aff},\lambda}(q^{-1})} \sum_{w \in W_{\mathrm{aff}}} w\left(e^\lambda \frac{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - e^{-\alpha}} \right)^{m_\alpha}.$$

Here $R_{+,\mathrm{aff}}$ is the set of positive coroots of G_{aff}, m_α is the multiplicity of α. This is wrong already for $\lambda = 0$! I.e.:

$$\mathcal{S}_{\mathrm{aff}}(h_0) = 1 \neq \Delta := \frac{1}{W_{\mathrm{aff}}(q^{-1})} \sum_{w \in W_{\mathrm{aff}}} w\left(\frac{\prod\limits_{\alpha \in R_{\mathrm{aff},+}} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_{\mathrm{aff},+}} 1 - e^{-\alpha}} \right)^{m_\alpha}.$$

Affine Macdonald formula

Want: Analog of Macdonald formula in the affine case.
The (topological) basis $\{h_\lambda\}$ of $\mathcal{H}_{\text{sph}}(G_{\text{aff}}, \mathcal{K})$ can be defined as in the finite-dimensional case. One might expect that

$$\mathcal{S}_{\text{aff}}(h_\lambda) = \frac{q^{\langle \lambda, \rho_{\text{aff}}^\vee \rangle}}{W_{\text{aff},\lambda}(q^{-1})} \sum_{w \in W_{\text{aff}}} w \left(e^\lambda \frac{\prod\limits_{\alpha \in R_{+,\text{aff}}} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_{+,\text{aff}}} 1 - e^{-\alpha}} \right)^{m_\alpha}.$$

Here $R_{+,\text{aff}}$ is the set of positive coroots of G_{aff}, m_α is the multiplicity of α. This is wrong already for $\lambda = 0$! I.e.:

$$\mathcal{S}_{\text{aff}}(h_0) = 1 \neq \Delta := \frac{1}{W_{\text{aff}}(q^{-1})} \sum_{w \in W_{\text{aff}}} w \left(\frac{\prod\limits_{\alpha \in R_{\text{aff},+}} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_{\text{aff},+}} 1 - e^{-\alpha}} \right)^{m_\alpha}.$$

Affine Macdonald formula

Want: Analog of Macdonald formula in the affine case.
The (topological) basis $\{h_\lambda\}$ of $\mathcal{H}_{\mathrm{sph}}(G_{\mathrm{aff}},\mathcal{K})$ can be defined as in the finite-dimensional case. One might expect that

$$\mathcal{S}_{\mathrm{aff}}(h_\lambda)=\frac{q^{\langle\lambda,\rho^\vee_{\mathrm{aff}}\rangle}}{W_{\mathrm{aff},\lambda}(q^{-1})}\sum_{w\in W_{\mathrm{aff}}} w\left(e^\lambda\frac{\prod\limits_{\alpha\in R_{+,\mathrm{aff}}} 1-q^{-1}e^{-\alpha}}{\prod\limits_{\alpha\in R_{+,\mathrm{aff}}} 1-e^{-\alpha}}\right)^{m_\alpha}.$$

Here $R_{+,\mathrm{aff}}$ is the set of positive coroots of G_{aff}, m_α is the multiplicity of α. This is wrong already for $\lambda=0$! I.e.:

$$\mathcal{S}_{\mathrm{aff}}(h_0)=1\neq\Delta:=\frac{1}{W_{\mathrm{aff}}(q^{-1})}\sum_{w\in W_{\mathrm{aff}}} w\left(\frac{\prod\limits_{\alpha\in R_{\mathrm{aff},+}} 1-q^{-1}e^{-\alpha}}{\prod\limits_{\alpha\in R_{\mathrm{aff},+}} 1-e^{-\alpha}}\right)^{m_\alpha}.$$

Theorem (Braverman-Kazhdan-Patnaik)

For every $\lambda \in \Lambda_{\mathrm{aff},+}$ we have

$$\mathcal{S}_{\mathrm{aff}}(h_\lambda) = \frac{1}{\Delta} \cdot \frac{q^{\langle \lambda, \rho_{\mathrm{aff}}^\vee \rangle}}{W_{\mathrm{aff},\lambda}(q^{-1})} \sum_{w \in W_{\mathrm{aff}}} w \left(e^{\lambda} \frac{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - e^{-\alpha}} \right)^{m_\alpha}$$

I.e. the formula is the same as in the affine case up to division by a "correction term" Δ! This is very curious - no good explanation is known.

Some partial geometric explanation was given in the work of Braverman, Kazhdan and Finkelberg - works only in the affine case (not for general Kac-Moody).

Theorem (Braverman-Kazhdan-Patnaik)

For every $\lambda \in \Lambda_{\mathrm{aff},+}$ *we have*

$$\mathcal{S}_{\mathrm{aff}}(h_\lambda) = \frac{1}{\Delta} \cdot \frac{q^{\langle \lambda, \rho_{\mathrm{aff}}^\vee \rangle}}{W_{\mathrm{aff},\lambda}(q^{-1})} \sum_{w \in W_{\mathrm{aff}}} w \left(e^{\lambda} \frac{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - e^{-\alpha}} \right)^{m_\alpha}$$

I.e. the formula is the same as in the affine case up to division by a "correction term" Δ! This is very curious - no good explanation is known.

Some partial geometric explanation was given in the work of Braverman, Kazhdan and Finkelberg - works only in the affine case (not for general Kac-Moody).

Theorem (Braverman-Kazhdan-Patnaik)

For every $\lambda \in \Lambda_{\mathrm{aff},+}$ we have

$$\mathcal{S}_{\mathrm{aff}}(h_\lambda) = \frac{1}{\Delta} \cdot \frac{q^{\langle \lambda, \rho_{\mathrm{aff}}^\vee \rangle}}{W_{\mathrm{aff},\lambda}(q^{-1})} \sum_{w \in W_{\mathrm{aff}}} w \left(e^\lambda \frac{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - q^{-1} e^{-\alpha}}{\prod\limits_{\alpha \in R_{+,\mathrm{aff}}} 1 - e^{-\alpha}} \right)^{m_\alpha}$$

I.e. the formula is the same as in the affine case up to division by a "correction term" Δ! This is very curious - no good explanation is known.

Some partial geometric explanation was given in the work of Braverman, Kazhdan and Finkelberg - works only in the affine case (not for general Kac-Moody).

Macdonald (using works of Cherednik): Δ has an infinite product formula. E.g. when G is simply laced we have

$$\Delta = \prod_{i=1}^{r} \prod_{j=1}^{\infty} \frac{1 - q^{-d_i} e^{-j\delta}}{1 - q^{-d_i+1} e^{-j\delta}}.$$

Here $d_1, ..., d_r$ are the exponents of G, δ - minimal imaginary root.
This formula is important for global application (constant term of Eisenstein series) - has to do with appearance of certain L-functions. It can be reproved using affine Satake + the work of Braverman, Kazhdan and Finkelberg.
It is not clear whether there exists a good formula for Δ for more general Kac-Moody groups - any example will be very interesting.

Macdonald (using works of Cherednik): Δ has an infinite product formula. E.g. when G is simply laced we have

$$\Delta = \prod_{i=1}^{r} \prod_{j=1}^{\infty} \frac{1 - q^{-d_i} e^{-j\delta}}{1 - q^{-d_i+1} e^{-j\delta}}.$$

Here $d_1, ..., d_r$ are the exponents of G, δ - minimal imaginary root. This formula is important for global application (constant term of Eisenstein series) - has to do with appearance of certain L-functions. It can be reproved using affine Satake + the work of Braverman, Kazhdan and Finkelberg.

It is not clear whether there exists a good formula for Δ for more general Kac-Moody groups - any example will be very interesting.

Macdonald (using works of Cherednik): Δ has an infinite product formula. E.g. when G is simply laced we have

$$\Delta = \prod_{i=1}^{r} \prod_{j=1}^{\infty} \frac{1 - q^{-d_i} e^{-j\delta}}{1 - q^{-d_i+1} e^{-j\delta}}.$$

Here $d_1, ..., d_r$ are the exponents of G, δ - minimal imaginary root. This formula is important for global application (constant term of Eisenstein series) - has to do with appearance of certain L-functions. It can be reproved using affine Satake + the work of Braverman, Kazhdan and Finkelberg.
It is not clear whether there exists a good formula for Δ for more general Kac-Moody groups - any example will be very interesting.

The Iwahori case

One can study the Hecke algebras for more general $K \subset G$.

Important example: The Iwahori group $I \subset G(\mathcal{O})$ (preimage of a Borel subgroup under the map $G(\mathcal{O}) \to G(\mathbb{k})$.

The algebra $\mathcal{H}(G, I)$ can be described by generators and relations; it is known as the *affine Hecke algebra* of G. It can be defined for any $q \in \mathbb{C}^*$ and for $q = 1$ it becomes $\mathbb{C}[W_{\text{aff}}]$.

One can consider $I_{\text{aff}} \subset G_{\text{aff}}(\mathcal{O})$ and the algebra $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is well-defined (no completion is needed!).

Braverman-Kazhdan-Patnaik: $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is isomorphic to a twisted version of Cherednik's DAHA (in particular, it is also defined for any $q \in \mathbb{C}^*$).

Interesting corollary: The above "twisted version" of DAHA has a basis parametrized by $I_{\text{aff}} \backslash G_{\text{aff}}(\mathcal{K})^+ / I_{\text{aff}}$. No algebraic description of this basis is known.

The Iwahori case

One can study the Hecke algebras for more general $K \subset G$.
Important example: The Iwahori group $I \subset G(\mathcal{O})$ (preimage of a Borel subgroup under the map $G(\mathcal{O}) \to G(\mathbb{k})$.
The algebra $\mathcal{H}(G, I)$ can be described by generators and relations; it is known as the *affine Hecke algebra* of G. It can be defined for any $q \in \mathbb{C}^*$ and for $q = 1$ it becomes $\mathbb{C}[W_{\text{aff}}]$.
One can consider $I_{\text{aff}} \subset G_{\text{aff}}(\mathcal{O})$ and the algebra $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is well-defined (no completion is needed!).
Braverman-Kazhdan-Patnaik: $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is isomorphic to a twisted version of Cherednik's DAHA (in particular, it is also defined for any $q \in \mathbb{C}^*$).
Interesting corollary: The above "twisted version" of DAHA has a basis parametrized by $I_{\text{aff}} \backslash G_{\text{aff}}(\mathcal{K})^+ / I_{\text{aff}}$. No algebraic description of this basis is known.

The Iwahori case

One can study the Hecke algebras for more general $K \subset G$.
Important example: The Iwahori group $I \subset G(\mathcal{O})$ (preimage of a Borel subgroup under the map $G(\mathcal{O}) \to G(\Bbbk)$.
The algebra $\mathcal{H}(G, I)$ can be described by generators and relations; it is known as the *affine Hecke algebra* of G. It can be defined for any $q \in \mathbb{C}^*$ and for $q = 1$ it becomes $\mathbb{C}[W_{\text{aff}}]$.
One can consider $I_{\text{aff}} \subset G_{\text{aff}}(\mathcal{O})$ and the algebra $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is well-defined (no completion is needed!).
Braverman-Kazhdan-Patnaik: $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is isomorphic to a twisted version of Cherednik's DAHA (in particular, it is also defined for any $q \in \mathbb{C}^*$).
Interesting corollary: The above "twisted version" of DAHA has a basis parametrized by $I_{\text{aff}} \backslash G_{\text{aff}}(\mathcal{K})^+ / I_{\text{aff}}$. No algebraic description of this basis is known.

The Iwahori case

One can study the Hecke algebras for more general $K \subset G$.
Important example: The Iwahori group $I \subset G(\mathcal{O})$ (preimage of a Borel subgroup under the map $G(\mathcal{O}) \to G(\mathbb{k})$.
The algebra $\mathcal{H}(G, I)$ can be described by generators and relations; it is known as the *affine Hecke algebra* of G. It can be defined for any $q \in \mathbb{C}^*$ and for $q = 1$ it becomes $\mathbb{C}[W_{\text{aff}}]$.
One can consider $I_{\text{aff}} \subset G_{\text{aff}}(\mathcal{O})$ and the algebra $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is well-defined (no completion is needed!).
Braverman-Kazhdan-Patnaik: $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is isomorphic to a twisted version of Cherednik's DAHA (in particular, it is also defined for any $q \in \mathbb{C}^*$).
Interesting corollary: The above "twisted version" of DAHA has a basis parametrized by $I_{\text{aff}} \backslash G_{\text{aff}}(\mathcal{K})^+ / I_{\text{aff}}$. No algebraic description of this basis is known.

The Iwahori case

One can study the Hecke algebras for more general $K \subset G$.

Important example: The Iwahori group $I \subset G(\mathcal{O})$ (preimage of a Borel subgroup under the map $G(\mathcal{O}) \to G(\mathbb{k})$.

The algebra $\mathcal{H}(G, I)$ can be described by generators and relations; it is known as the *affine Hecke algebra* of G. It can be defined for any $q \in \mathbb{C}^*$ and for $q = 1$ it becomes $\mathbb{C}[W_{\text{aff}}]$.

One can consider $I_{\text{aff}} \subset G_{\text{aff}}(\mathcal{O})$ and the algebra $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is well-defined (no completion is needed!).

Braverman-Kazhdan-Patnaik: $\mathcal{H}(G_{\text{aff}}(\mathcal{K})^+, I_{\text{aff}})$ is isomorphic to a twisted version of Cherednik's DAHA (in particular, it is also defined for any $q \in \mathbb{C}^*$).

Interesting corollary: The above "twisted version" of DAHA has a basis parametrized by $I_{\text{aff}} \backslash G_{\text{aff}}(\mathcal{K})^+ / I_{\text{aff}}$. No algebraic description of this basis is known.

Geometric Satake

The Satake isomorphism has a geometric counterpart, called *the geometric Satake isomorphism*.

Namely, replace $calK = \mathbb{C}((s))$ (and $\mathcal{O} = \mathbb{C}[[s]]$); set $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ (the affine Grassmannian).

Theorem (Geometric Satake isomorphism)

The tensor category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr *is equivalent to* $\mathrm{Rep}(G^\vee)$.

A partial generalization of this equivalence (and various constructions around it) to the affine case was given in the paper of Braverman and Finkelberg; geometry of moduli spaces of G-bundles on algebraic surfaces plays an imporatant role there!

Geometric Satake

The Satake isomorphism has a geometric counterpart, called *the geometric Satake isomorphism*.
Namely, replace *calK* $= \mathbb{C}((s))$ (and $\mathcal{O} = \mathbb{C}[[s]]$); set $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ (the affine Grassmannian).

Theorem (Geometric Satake isomorphism)

The tensor category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr *is equivalent to* $\mathrm{Rep}(G^\vee)$.

A partial generalization of this equivalence (and various constructions around it) to the affine case was given in the paper of Braverman and Finkelberg; geometry of moduli spaces of G-bundles on algebraic surfaces plays an imporatant role there!

Geometric Satake

The Satake isomorphism has a geometric counterpart, called *the geometric Satake isomorphism*.
Namely, replace *calK* $= \mathbb{C}((s))$ (and $\mathcal{O} = \mathbb{C}[[s]]$); set $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ (the affine Grassmannian).

Theorem (Geometric Satake isomorphism)

The tensor category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr *is equivalent to* $\mathrm{Rep}(G^\vee)$.

A partial generalization of this equivalence (and various constructions around it) to the affine case was given in the paper of Braverman and Finkelberg; geometry of moduli spaces of *G*-bundles on algebraic surfaces plays an imporatant role there!

Unramified automorphic forms over functional fields

We shall now discuss some results for global fields. We'll be very sketchy!

Let X be a smooth projective geometrically irreducible curve over a finite filed $\mathbb{k} = \mathbb{F}_q$. Let also G be a split semi-simple simply connected group over $\mathbb{k}$. We set $F = \mathbb{k}(X)$; this is a global field and we let $\mathbb{A}_F$ denote its ring of adeles. We also denote by $\mathcal{O}(\mathbb{A}_F)$ ring of integral adeles.

It is well-known that the double quotient

$$G(\mathcal{O}(\mathbb{A}_F))\backslash G(\mathbb{A}_F)/G(F) \simeq \mathrm{Bun}_G(X).$$

Here $\mathrm{Bun}_G(X)$ denotes the set of $\mathbb{F}_q$-points of the moduli space of G-bundles on X (in practice, it is often convenient to treat it as a groupoid; the above isomorphism can then be upgraded to an equivalence of groupoids).

(Complex valued) functions on $\mathrm{Bun}_G(X)$ will usually be referred to as unramified automorphic forms. We denote the space of such functions by $\mathbb{C}(\mathrm{Bun}_G(X))$.

Unramified automorphic forms over functional fields

We shall now discuss some results for global fields. We'll be very sketchy!

Let X be a smooth projective geometrically irreducible curve over a finite filed $\Bbbk = \mathbb{F}_q$. Let also G be a split semi-simple simply connected group over $\Bbbk$. We set $F = \Bbbk(X)$; this is a global field and we let $\mathbb{A}_F$ denote its ring of adeles. We also denote by $\mathcal{O}(\mathbb{A}_F)$ ring of integral adeles.

It is well-known that the double quotient

$$G(\mathcal{O}(\mathbb{A}_F))\backslash G(\mathbb{A}_F)/G(F) \simeq \mathrm{Bun}_G(X).$$

Here $\mathrm{Bun}_G(X)$ denotes the set of $\mathbb{F}_q$-points of the moduli space of G-bundles on X (in practice, it is often convenient to treat it as a groupoid; the above isomorphism can then be upgraded to an equivalence of groupoids).

(Complex valued) functions on $\mathrm{Bun}_G(X)$ will usually be referred to as unramified automorphic forms. We denote the space of such functions by $\mathbb{C}(\mathrm{Bun}_G(X))$.

Unramified automorphic forms over functional fields

We shall now discuss some results for global fields. We'll be very sketchy!

Let X be a smooth projective geometrically irreducible curve over a finite filed $\mathbb{k} = \mathbb{F}_q$. Let also G be a split semi-simple simply connected group over $\mathbb{k}$. We set $F = \mathbb{k}(X)$; this is a global field and we let $\mathbb{A}_F$ denote its ring of adeles. We also denote by $\mathcal{O}(\mathbb{A}_F)$ ring of integral adeles.

It is well-known that the double quotient

$$G(\mathcal{O}(\mathbb{A}_F))\backslash G(\mathbb{A}_F)/G(F) \simeq \mathrm{Bun}_G(X).$$

Here $\mathrm{Bun}_G(X)$ denotes the set of $\mathbb{F}_q$-points of the moduli space of G-bundles on X (in practice, it is often convenient to treat it as a groupoid; the above isomorphism can then be upgraded to an equivalence of groupoids).

(Complex valued) functions on $\mathrm{Bun}_G(X)$ will usually be referred to as unramified automorphic forms. We denote the space of such functions by $\mathbb{C}(\mathrm{Bun}_G(X))$.

L-functions

Let v be a place of F and let $\mathcal{K}_v$ be the corresponding local completion of F with ring of integers $\mathcal{O}_v$. Let q_v be the number of elements in the residue field of $\mathcal{O}_v$. Let $\mathcal{H}_v(G)$ denote the corresponding Hecke algebra. It is well-known that $\mathcal{H}_v(G)$ acts on $\mathbb{C}(\mathrm{Bun}_G(X))$ by correspondences.

Assume that $f \in \mathbb{C}(\mathrm{Bun}_G(X))$ is an eigen-function of all the $\mathcal{H}_v(G)$. The corresponding eigen-value is an element $g_v(f) \in G^\vee$ for all places v. Given a finite-dimensional representation $\rho : G^\vee \to \mathrm{Aut}(V)$, define by

$$L_G(f, \rho, s) = \prod_v \frac{1}{\det(1 - q_v^{-s}\rho(g_v))}.$$

The product converges for $\mathrm{Re}(s) >> 0$.

L-functions

Let v be a place of F and let $\mathcal{K}_v$ be the corresponding local completion of F with ring of integers $\mathcal{O}_v$. Let q_v be the number of elements in the residue field of $\mathcal{O}_v$. Let $\mathcal{H}_v(G)$ denote the corresponding Hecke algebra. It is well-known that $\mathcal{H}_v(G)$ acts on $\mathbb{C}(\mathrm{Bun}_G(X))$ by correspondences.

Assume that $f \in \mathbb{C}(\mathrm{Bun}_G(X))$ is an eigen-function of all the $\mathcal{H}_v(G)$. The corresponding eigen-value is an element $g_v(f) \in G^\vee$ for all places v. Given a finite-dimensional representation $\rho : G^\vee \to \mathrm{Aut}(V)$, define by

$$L_G(f, \rho, s) = \prod_v \frac{1}{\det(1 - q_v^{-s}\rho(g_v))}.$$

The product converges for $\mathrm{Re}(s) >> 0$.

Eisenstein series

Langlands proved meromorphic continuation for $L(f, \rho, s)$ in a number of cases using the theory of Eisenstein series.
Let $P \subset G$ be parabolic subgroup and let M be the corresponding Levi group. We have canonical maps $G \leftarrow P \rightarrow M$ which give rise to the diagram

$$\begin{array}{ccc} \mathrm{Bun}_P(X) & \xrightarrow{\eta} & \mathrm{Bun}_M(X) \\ {\scriptstyle\pi}\big\downarrow & & \\ \mathrm{Bun}_G(X) & & \end{array}$$

Given a function $f \in \mathbb{C}(\mathrm{Bun}_M(X))$ we define the Eisenstein series $\mathrm{Eis}_{GP}(f) \in \mathbb{C}(\mathrm{Bun}_G(X))$ by setting

$$\mathrm{Eis}_{GP}(f) = \pi_! \eta^*(f).$$

$\mathrm{Eis}_{GP}(f)$ makes sense when f has finite support.
What happens for other functions?

The connected components of Bun_M are numbered by elements of the lattice $\Lambda_M = \mathrm{Hom}(\mathbb{G}_m, M/[M, M])$; hence any $\mathfrak{s} \in \mathrm{Hom}(\Lambda_M, \mathbb{C})$ defines a function on Bun_M whose value on the connected component corresponding to $\theta \in \Lambda_M$ is $q^{\mathfrak{s}(\theta)}$. For any any function f on Bun_M we denote by $f_{\mathfrak{s}}$ the product of f and this function.

Theorem (Langlands)

Assume that f is a Hecke eigen-form then $\mathrm{Eis}_{GP}(f_{\mathfrak{s}})$ *is well-defined as a meromorphic function of* $\mathfrak{s}$.

$\mathrm{Eis}_{GP}(f)$ makes sense when f has finite support.
What happens for other functions?
The connected components of Bun_M are numbered by elements of the lattice $\Lambda_M = \mathrm{Hom}(\mathbb{G}_m, M/[M,M])$; hence any $\mathfrak{s} \in \mathrm{Hom}(\Lambda_M, \mathbb{C})$ defines a function on Bun_M whose value on the connected component corresponding to $\theta \in \Lambda_M$ is $q^{\mathfrak{s}(\theta)}$. For any any function f on Bun_M we denote by $f_{\mathfrak{s}}$ the product of f and this function.

Theorem (Langlands)

Assume that f is a Hecke eigen-form then $\mathrm{Eis}_{GP}(f_{\mathfrak{s}})$ is well-defined as a meromorphic function of $\mathfrak{s}$.

$\text{Eis}_{GP}(f)$ makes sense when f has finite support.
What happens for other functions?
The connected components of Bun_M are numbered by elements of the lattice $\Lambda_M = \text{Hom}(\mathbb{G}_m, M/[M, M])$; hence any $\mathfrak{s} \in \text{Hom}(\Lambda_M, \mathbb{C})$ defines a function on Bun_M whose value on the connected component corresponding to $\theta \in \Lambda_M$ is $q^{\mathfrak{s}(\theta)}$. For any any function f on Bun_M we denote by $f_{\mathfrak{s}}$ the product of f and this function.

Theorem (Langlands)

Assume that f is a Hecke eigen-form then $\text{Eis}_{GP}(f_{\mathfrak{s}})$ *is well-defined as a meromorphic function of* $\mathfrak{s}$.

Constant term and L-functions

For a function $g \in \mathbb{C}(\mathrm{Bun}_G(X))$ we define the constant term $c_{GP}(g) \in \mathbb{C}(\mathrm{Bun}_M(X))$ by

$$c_{GP}(g) = \eta_! \pi^*(g).$$

The constant term is well-defined for any g. We say that g is cuspidal if $c_{GP}(g) = 0$ for all P.

Let P_- be a parabolic subgroup opposite to P. Let $M^\vee$ be the corresponding Langlands dual group and let also $\mathfrak{n}_P^\vee$, be the nilpotent radical of the Lie algebra of $P^\vee$.

Langlands: For a Hecke eigen-form f the function $c_{GP_-} \circ \mathrm{Eis}_{GP}(f_s)$ can be expressed via $L(f, \rho, s)$ where ρ appears in $\mathfrak{n}_P^\vee$.

Langlands used the above results in order to prove meromorphic continuation for such L-functions.

Constant term and L-functions

For a function $g \in \mathbb{C}(\mathrm{Bun}_G(X))$ we define the constant term $c_{GP}(g) \in \mathbb{C}(\mathrm{Bun}_M(X))$ by

$$c_{GP}(g) = \eta_! \pi^*(g).$$

The constant term is well-defined for any g. We say that g is cuspidal if $c_{GP}(g) = 0$ for all P.
Let P_- be a parabolic subgroup opposite to P. Let $M^\vee$ be the corresponding Langlands dual group and let also $\mathfrak{n}_P^\vee$, be the nilpotent radical of the Lie algebra of $P^\vee$.
Langlands: For a Hecke eigen-form f the function $c_{GP_-} \circ \mathrm{Eis}_{GP}(f_s)$ can be expressed via $L(f, \rho, s)$ where ρ appears in $\mathfrak{n}_P^\vee$.
Langlands used the above results in order to prove meromorphic continuation for such L-functions.

Constant term and L-functions

For a function $g \in \mathbb{C}(\mathrm{Bun}_G(X))$ we define the constant term $c_{GP}(g) \in \mathbb{C}(\mathrm{Bun}_M(X))$ by

$$c_{GP}(g) = \eta_! \pi^*(g).$$

The constant term is well-defined for any g. We say that g is cuspidal if $c_{GP}(g) = 0$ for all P.
Let P_- be a parabolic subgroup opposite to P. Let $M^\vee$ be the corresponding Langlands dual group and let also $\mathfrak{n}_P^\vee$, be the nilpotent radical of the Lie algebra of $P^\vee$.
Langlands: For a Hecke eigen-form f the function $c_{GP_-} \circ \mathrm{Eis}_{GP}(f_\mathfrak{s})$ can be expressed via $L(f, \rho, s)$ where ρ appears in $\mathfrak{n}_P^\vee$.
Langlands used the above results in order to prove meromorphic continuation for such L-functions.

Constant term and L-functions

For a function $g \in \mathbb{C}(\mathrm{Bun}_G(X))$ we define the constant term $c_{GP}(g) \in \mathbb{C}(\mathrm{Bun}_M(X))$ by

$$c_{GP}(g) = \eta_! \pi^*(g).$$

The constant term is well-defined for any g. We say that g is cuspidal if $c_{GP}(g) = 0$ for all P.
Let P_- be a parabolic subgroup opposite to P. Let $M^\vee$ be the corresponding Langlands dual group and let also $\mathfrak{n}_P^\vee$, be the nilpotent radical of the Lie algebra of $P^\vee$.
Langlands: For a Hecke eigen-form f the function $c_{GP_-} \circ \mathrm{Eis}_{GP}(f_{\mathfrak{s}})$ can be expressed via $L(f, \rho, s)$ where ρ appears in $\mathfrak{n}_P^\vee$.
Langlands used the above results in order to prove meromorphic continuation for such L-functions.

Conclusion:

Langlands' method works if our group can be realized as a Levi subgroup in a bigger group and the reprsentation ρ appears in the corresponding nilpotent radical.

This is pretty rare!

Idea: Kac-Moody groups provide additional source of such examples. For example, for any G the group G_{aff} contains $G \times \mathbb{G}_m$ as a Levi subgroup.

Conclusion:

Langlands' method works if our group can be realized as a Levi subgroup in a bigger group and the reprsentation ρ appears in the corresponding nilpotent radical.
This is pretty rare!
Idea: Kac-Moody groups provide additional source of such examples. For example, for any G the group G_{aff} contains $G \times \mathbb{G}_m$ as a Levi subgroup.

Conclusion:

Langlands' method works if our group can be realized as a Levi subgroup in a bigger group and the reprsentation ρ appears in the corresponding nilpotent radical.
This is pretty rare!
Idea: Kac-Moody groups provide additional source of such examples.
For example, for any G the group G_{aff} contains $G \times \mathbb{G}_m$ as a Levi subgroup.

Unramfied automorphic forms for G_{aff}

It is better to work with $\widehat{G}_{\text{aff}}$ (completed version).
How to think about $\text{Bun}_{\widehat{G}_{\text{aff}}}(X)$?

Since we are given a homomorphism $\widehat{G}_{\text{aff}} \to \mathbb{G}_m$, if such a notion makes sense, then we should have a functor

$$\iota : \text{Bun}_{\widehat{G}_{\text{aff}}} \to \text{Pic}(X).$$

So, in order to describe $\text{Bun}_{\widehat{G}_{\text{aff}}}$, it is enough to describe the groupoid $\eta^{-1}(\mathcal{L})$ for each $\mathcal{L} \in \text{Pic}(X)$ in a way that the assignment $\mathcal{L} \mapsto \eta^{-1}(\mathcal{L})$ is functorial with respect to isomorphisms in $\text{Pic}(X)$.

Unramfied automorphic forms for G_{aff}

It is better to work with $\widehat{G}_{\text{aff}}$ (completed version).
How to think about $\text{Bun}_{\widehat{G}_{\text{aff}}}(X)$?

Since we are given a homomorphism $\widehat{G}_{\text{aff}} \to \mathbb{G}_m$, if such a notion makes sense, then we should have a functor

$$\iota : \text{Bun}_{\widehat{G}_{\text{aff}}} \to \text{Pic}(X).$$

So, in order to describe $\text{Bun}_{\widehat{G}_{\text{aff}}}$, it is enough to describe the groupoid $\eta^{-1}(\mathcal{L})$ for each $\mathcal{L} \in \text{Pic}(X)$ in a way that the assignment $\mathcal{L} \mapsto \eta^{-1}(\mathcal{L})$ is functorial with respect to isomorphisms in $\text{Pic}(X)$.

Let $S_{\mathcal{L}}$ be the total space of $\mathcal{L}$ and $\widehat{S}_{\mathcal{L}}$ be its formal completion along zero section.
We can consider $\mathrm{Bun}_G(\widehat{S}^0_{\mathcal{L}})$, where $\widehat{S}^0_{\mathcal{L}} = \widehat{S}_{\mathcal{L}} \backslash X$ (this has to be defined carefully!).

Then $\iota^{-1}(\mathcal{L})$ is a certain ("determinant") Pic(X)-torsor $\widetilde{\mathrm{Bun}}_G(\widehat{S}^0)$ over $\mathrm{Bun}_G(\widehat{S}^0)$ (its definition is also closely related to the notion of 2nd Chern class).
Our automorphic forms (e.g. Eisenstein series) are going to be functions on $\widetilde{\mathrm{Bun}}_G(\widehat{S}^0_{\mathcal{L}})$ (usually $\mathcal{L}$ will be fixed).

Let $S_{\mathcal{L}}$ be the total space of $\mathcal{L}$ and $\widehat{S}_{\mathcal{L}}$ be its formal completion along zero section.
We can consider $\mathrm{Bun}_G(\widehat{S}^0_{\mathcal{L}})$, where $\widehat{S}^0_{\mathcal{L}} = \widehat{S}_{\mathcal{L}} \backslash X$ (this has to be defined carefully!).

Then $\iota^{-1}(\mathcal{L})$ is a certain ("determinant") Pic(X)-torsor $\widetilde{\mathrm{Bun}}_G(\widehat{S}^0)$ over $\mathrm{Bun}_G(\widehat{S}^0)$ (its definition is also closely related to the notion of 2nd Chern class).
Our automorphic forms (e.g. Eisenstein series) are going to be functions on $\widetilde{\mathrm{Bun}}_G(\widehat{S}^0_{\mathcal{L}})$ (usually $\mathcal{L}$ will be fixed).

The theory of affine Eisenstein series was initiated by Garland and Kapranov. Let us take

$$P_{\text{aff}} = \mathbb{G}_m \times G((t)) \rtimes \mathbb{G}_m; \quad P_{\text{aff},-} = \mathbb{G}_m \times G[t^{-1}] \rtimes \mathbb{G}_m.$$

Also, fix $\mathcal{L}$ such that $\deg \mathcal{L} < 0$.

It turns out that one can define the corresponding Eisenstein series and constant term. I.e. we can talk about operators $\mathrm{Eis}_{\mathcal{L}}, \mathrm{Eis}_{\mathcal{L}}^{-}$ acting from functions with finite support on $\mathrm{Bun}_G(X) \times \mathrm{Pic}(X)$ to functions on $\widetilde{\mathrm{Bun}}_G(\widehat{S}^0_{\mathcal{L}})$.

Also we can talk about $c_{\mathcal{L}}$ and $c_{\mathcal{L}}^{-}$ acting in the opposite direction.

Eisenstein series

The theory of affine Eisenstein series was initiated by Garland and Kapranov. Let us take

$$P_{\mathrm{aff}} = \mathbb{G}_m \times G((t)) \rtimes \mathbb{G}_m; \quad P_{\mathrm{aff},-} = \mathbb{G}_m \times G[t^{-1}] \rtimes \mathbb{G}_m.$$

Also, fix $\mathcal{L}$ such that $\deg \mathcal{L} < 0$.
It turns out that one can define the corresponding Eisenstein series and constant term. I.e. we can talk about operators $\mathrm{Eis}_{\mathcal{L}}, \mathrm{Eis}_{\mathcal{L}}^{-}$ acting from functions with finite support on $\mathrm{Bun}_G(X) \times \mathrm{Pic}(X)$ to functions on $\widetilde{\mathrm{Bun}}_G(\widehat{S}^0_{\mathcal{L}})$.
Also we can talk about $c_{\mathcal{L}}$ and $c_{\mathcal{L}}^{-}$ acting in the opposite direction.

Eisenstein series

The theory of affine Eisenstein series was initiated by Garland and Kapranov. Let us take

$$P_{\mathrm{aff}} = \mathbb{G}_m \times G((t)) \rtimes \mathbb{G}_m; \quad P_{\mathrm{aff},-} = \mathbb{G}_m \times G[t^{-1}] \rtimes \mathbb{G}_m.$$

Also, fix $\mathcal{L}$ such that $\deg \mathcal{L} < 0$.
It turns out that one can define the corresponding Eisenstein series and constant term. I.e. we can talk about operators $\mathrm{Eis}_{\mathcal{L}}, \mathrm{Eis}_{\mathcal{L}}^-$ acting from functions with finite support on $\mathrm{Bun}_G(X) \times \mathrm{Pic}(X)$ to functions on $\widetilde{\mathrm{Bun}}_G(\widehat{S}^0_{\mathcal{L}})$.
Also we can talk about $c_{\mathcal{L}}$ and $c_{\mathcal{L}}^-$ acting in the opposite direction.

Convergence issues

Can we define the above operators for Hecke eigen-functions?

It turns out that $\mathrm{Eis}_{\mathcal{L}}(f)$ is convergent for any cuspidal f (quite different from the usual case!) Also $c_{\mathcal{L}}(g)$ is convergent for any g.

The situation is trickier for $\mathrm{Eis}_{\mathcal{L}}^{-}$ and $c_{\mathcal{L}}^{-}$. Let f be a function on $\mathrm{Bun}_G(X)$. For any $(\mathcal{M},\mathcal{E}) \in \mathrm{Bun}_G(X) \times \mathrm{Pic}(X)$ and $s \in \mathbb{C}$ set

$$f_s(\mathcal{M},\mathcal{E}) = f(\mathcal{M}) \cdot q^{s \deg \mathcal{E}}.$$

Theorem (Braverman-Kazhdan)

Assume that f is cuspidal. Then $\mathrm{Eis}_{\mathcal{L}}^{-}(f_s)$ is convergent when $\mathrm{Re}(s) >> 0$.

There is a similar statement for $c_{\mathcal{L}}^{-}$.

Convergence issues

Can we define the above operators for Hecke eigen-functions?
It turns out that $\mathrm{Eis}_{\mathcal{L}}(f)$ is convergent for any cuspidal f (quite different from the usual case!) Also $c_{\mathcal{L}}(g)$ is convergent for any g.
The situation is trickier for $\mathrm{Eis}_{\mathcal{L}}^{-}$ and $c_{\mathcal{L}}^{-}$. Let f be a function on $\mathrm{Bun}_G(X)$. For any $(\mathcal{M},\mathcal{E}) \in \mathrm{Bun}_G(X) \times \mathrm{Pic}(X)$ and $s \in \mathbb{C}$ set

$$f_s(\mathcal{M},\mathcal{E}) = f(\mathcal{M}) \cdot q^{s \deg \mathcal{E}}.$$

Theorem (Braverman-Kazhdan)

Assume that f is cuspidal. Then $\mathrm{Eis}_{\mathcal{L}}^{-}(f_s)$ is convergent when $\mathrm{Re}(s) >> 0$.

There is a similar statement for $c_{\mathcal{L}}^{-}$.

Convergence issues

Can we define the above operators for Hecke eigen-functions?
It turns out that $\mathrm{Eis}_{\mathcal{L}}(f)$ is convergent for any cuspidal f (quite different from the usual case!) Also $c_{\mathcal{L}}(g)$ is convergent for any g.
The situation is trickier for $\mathrm{Eis}_{\mathcal{L}}^{-}$ and $c_{\mathcal{L}}^{-}$. Let f be a function on $\mathrm{Bun}_G(X)$. For any $(\mathcal{M}, \mathcal{E}) \in \mathrm{Bun}_G(X) \times \mathrm{Pic}(X)$ and $s \in \mathbb{C}$ set

$$f_s(\mathcal{M}, \mathcal{E}) = f(\mathcal{M}) \cdot q^{s \deg \mathcal{E}}.$$

Theorem (Braverman-Kazhdan)

Assume that f is cuspidal. Then $\mathrm{Eis}_{\mathcal{L}}^{-}(f_s)$ is convergent when $\mathrm{Re}(s) >> 0$.

There is a similar statement for $c_{\mathcal{L}}^{-}$.

Relation to L-functions

Theorem

Let f again be a cuspidal Hecke eigen-function on $\mathrm{Bun}_G(X)$. Then the functions $c_{\mathcal{L}}^{-} \circ \mathrm{Eis}_{\mathcal{L}}(f_s)$ and $c_{\mathcal{L}} \circ \mathrm{Eis}_{\mathcal{L}}^{-}(f_s)$ (both are well-defined in some half-plane) are proportional to f_s.
The coefficient is expressible through an infinite product of L-functions of the form $L(f, \rho, s)$.

In order to prove the above theorem one needs to use (a special case of) the affine Macdonald formula. In particular, the fact the "correction term" Δ has a product expansion is very important - without it we wouldn't be abel to express the answer only in terms of L-functions.

Relation to L-functions

Theorem

Let f again be a cuspidal Hecke eigen-function on $\mathrm{Bun}_G(X)$. Then the functions $c^-_{\mathcal{L}} \circ \mathrm{Eis}_{\mathcal{L}}(f_s)$ and $c_{\mathcal{L}} \circ \mathrm{Eis}^-_{\mathcal{L}}(f_s)$ (both are well-defined in some half-plane) are proportional to f_s.
The coefficient is expressible through an infinite product of L-functions of the form $L(f, \rho, s)$.

In order to prove the above theorem one needs to use (a special case of) the affine Macdonald formula. In particular, the fact the "correction term" Δ has a product expansion is very important - without it we wouldn't be abel to express the answer only in terms of L-functions.

What kind of L-functions?

Typical infinite product appearing in the Theorem is of the form

$$\prod_{n=1}^{\infty} L(ns),$$

where $L(s)$ is some usual automorphic L-function. It should be meromorphic for $\mathrm{Re}(s) > 0$ but should NOT have a continuation to any larger domain.
What kind of L? Assume that G is simply laced. Then the L's for which the above infinite product will appear will be either abelian or $L(f, \rho, s)$ where ρ is the adjoint representation. It is not known that such $L(f, \rho, s)$ is meromorphic for $\mathrm{Re}(s) > 0$!

A conjecture

The above discussion shows that $\mathrm{Eis}^{-}_{\mathcal{L}}(f_s)$ can be meromorphic only in some half plane. In a suitable normalization it converges for $\mathrm{Re}(s) > h^{\vee}$ (the dual Coxeter number) and should have meromorphic continuation to $\mathrm{Re}(s) > 0$.

This is a non-trivial conjecture! It implies, for example, the meromorphic continuation of the corresponding L-function.

There is also a conjectural analog of the functional equation for Eisenstein series in this context - will not discuss this.

A conjecture

The above discussion shows that $\mathrm{Eis}^{-}_{\mathcal{L}}(f_s)$ can be meromorphic only in some half plane. In a suitable normalization it converges for $\mathrm{Re}(s) > h^{\vee}$ (the dual Coxeter number) and should have meromorphic continuation to $\mathrm{Re}(s) > 0$.

This is a non-trivial conjecture! It implies, for example, the meromorphic continuation of the corresponding L-function.

There is also a conjectural analog of the functional equation for Eisenstein series in this context - will not discuss this.

A conjecture

The above discussion shows that $\mathrm{Eis}^{-}_{\mathcal{L}}(f_s)$ can be meromorphic only in some half plane. In a suitable normalization it converges for $\mathrm{Re}(s) > h^{\vee}$ (the dual Coxeter number) and should have meromorphic continuation to $\mathrm{Re}(s) > 0$.

This is a non-trivial conjecture! It implies, for example, the meromorphic continuation of the corresponding L-function.

There is also a conjectural analog of the functional equation for Eisenstein series in this context - will not discuss this.