Dynamics of non-archimedean Polish groups

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Kraków July 6, 2012

Introduction

A Polish group is a topological group whose topology is Polish, i.e., induced by a compatible complete, separable metric. Such a group is non-archimedean if it has a nbhd basis at the identity consisting of open subgroups.

In recent times there has been considerable activity in the study of the dynamics of these groups and this work has led to interesting interactions between logic, combinatorics, group theory (both in the topological and algebraic context), topological dynamics, ergodic theory and representation theory. In this lecture I will give a bird's eye view of some aspects of this area of research, concentrating on the main directions as opposed to a detailed exposition of individual results .

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Introduction

The main directions of research in this area are in:

- Topological dynamics
- Ergodic theory
- Unitary representations

Automorphism groups

From the point of view of logic non-archimedean Polish groups can be viewed as automorphism groups of countable structures and I will first describe this point of view and the necessary background.

Automorphism groups

Definition

A structure $\boldsymbol{A} = \langle A, f, g, \ldots, R, S, \ldots \rangle$ is a nonempty set A together with families of distinguished functions (of several variables) with arguments and values in A, relations (of several arguments) on A. In this lecture, I will always assume that there only countably many such functions and relations.

The sequence

$$(\mathrm{arity}(f), \mathrm{arity}(g), \ldots, \mathrm{arity}(R), \mathrm{arity}(S), \ldots, \ldots)$$

is called the signature of the structure $\boldsymbol{A}$.

Automorphism groups

Examples

- linear orders: $\boldsymbol{L} = \langle L, < \rangle$
- graphs: $\boldsymbol{G} = \langle G, E \rangle$
- groups: $\boldsymbol{H} = \langle H, \cdot, 1 \rangle$
- vector spaces over a field F: $\boldsymbol{V} = \langle V, +, f_a \rangle_{a \in F}$
- metric spaces: $\boldsymbol{X} = \langle X, R_q \rangle_{q \in \mathbb{Q}}$

Definition

A structure $\boldsymbol{A}$ as above is countable (resp., finite) if the set A is countable (resp., finite).

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Fraïssé structures

Certain countable structures play a crucial role in this theory.

Definition

A countable structure K is a Fraïssé structure if it satisfies the following properties:

- It is infinite.
- It is locally finite.
- It is ultrahomogeneous (i.e., an isomorphism between finite substructures can be extended to an automorphism of the whole structure).

Examples

- $\langle \mathbb{Q}, < \rangle$.
- The random graph.
- The (countable) atomless Boolean algebra.
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Definition

The age, $\mathrm{Age}(\boldsymbol{K})$, of a Fraïssé structure $\boldsymbol{K}$ is the family of all finite structures that can be embedded into it.

Definition

A class $\mathcal{K}$ of finite structures of the same signature is called a Fraïssé class if it satisfies the following properties:

- (HP) Hereditary property.
- (JEP) Joint embedding property.
- (AP) Amalgamation property.
- It is countable (up to $\cong$).
- It is unbounded.

It is easy to check that $\mathrm{Age}(\boldsymbol{K})$ is a Fraïssé class.

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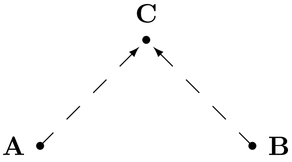
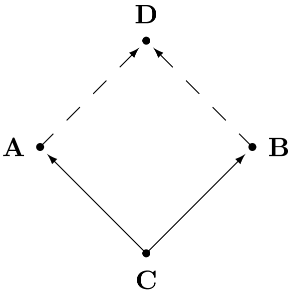
Joint embedding property (JEP)



Amalgamation property (AP)



Fraïssé structures

Conversely, Fraïssé showed that one can associate to each Fraïssé class $\mathcal{K}$ a canonical Fraïssé structure $\boldsymbol{K} = \mathrm{Frlim}(\mathcal{K})$, called its Fraïssé limit, which is the unique Fraïssé structure whose age is equal to $\mathcal{K}$ and therefore one has a canonical one-to-one correspondence:

$$\mathcal{K} \mapsto \mathrm{Frlim}(\mathcal{K})$$

between Fraïssé classes and Fraïssé structures whose inverse is:

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Examples

- finite graphs $\rightleftarrows$ random graph
- finite linear orderings $\rightleftarrows$ $\langle \mathbb{Q}, < \rangle$
- f.d. vector spaces $\rightleftarrows$ (countable) infinite-dimensional vector space (over a finite field)
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$\mathrm{Aut}(\boldsymbol{A})$ as a topological group

For a countable structure $\boldsymbol{A}$, we view $\mathrm{Aut}(\boldsymbol{A})$ as a Polish group with the pointwise convergence topology. We now have the following characterization of non-archimedean groups:

Theorem

For any Polish group G, the following are equivalent:

- *G is non-Archimedean.*
- *G is isomorphic to a closed subgroup of S_∞, the permutation group of $\mathbb{N}$ with the pointwise convergence topology.*
- *$G \cong \mathrm{Aut}(\boldsymbol{A})$, for a countable structure $\boldsymbol{A}$.*
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Dynamics of $\mathrm{Aut}(\boldsymbol{A})$

We will see how the study of the dynamics of these automorphism groups is connected with finite combinatorics, group theory (topological and algebraic), topological dynamics, ergodic theory and representation theory.

Part I. Topological dynamics: Universal minimal flows and structural Ramsey theory

Universal minimal flows

Below G is a (Hausdorff) topological group. A G-flow is a continuous action of G on a (Hausdorff, nonempty) compact space X. A subflow of X is a compact invariant set with the restriction of the action. A flow is minimal if there are no proper subflows or equivalently every orbit is dense. Every G-flow contains a minimal subflow. A homomorphism between two G-flows X, Y is a continuous G-map $\pi : X \to Y$. If Y is minimal, then π must be onto. An isomorphism is a bijective homomorphism.

Theorem

For any G, there is a minimal G-flow, $M(G)$, called its universal minimal flow with the following property: For any minimal G-flow X, there is a homomorphism $\pi : M(G) \to X$. Moreover $M(G)$ is uniquely determined up to isomorphism by this property.

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Universal minimal flows

If G is compact, then $M(G) = G$. If G is non-compact but locally compact, then $M(G)$ is extremely complicated, e.g., it is non-metrizable. However, it is a remarkable phenomenon that for non-locally compact groups G, $M(G)$ can even trivialize (i.e., can be a singleton)!

This leads to two general problems in topological dynamics:

- When is $M(G)$ trivial?
- Even if it is not trivial, can one explicitly determine $M(G)$ and show that it is metrizable?

There has been an extensive study of these problems in the last 30 years or so in the work of Gromov, Milman, Pestov, Glasner, Weiss, Giordano, Uspenskii and others.

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Extreme amenability

Definition

A group G is called extremely amenable if its universal minimal flow $M(G)$ is trivial.

This is equivalent to saying that G has an extremely strong fixed point property: Every G-flow has a fixed point. For that reason, sometimes extremely amenable groups are also said to have the fixed point on compacta property.

T. Mitchell (1966) raised the question of their existence. Granirer-Lau and Veech showed in the 1970's that no non-trivial locally compact group can be extremely amenable. The first examples of extremely amenable groups were produced by Herer-Christensen (1975), who, apparently unaware of Mitchell's question, showed that there are Polish abelian groups that are "exotic", i.e., admit no non-trivial unitary representations. Such groups are extremely amenable.

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- Furstenberg-Weiss, Glasner (1998): $L(X, \mu, \mathbb{T})$.
- Pestov (2002): $\mathrm{Iso}(\mathbb{U})$.
- Giordano-Pestov (2002): $\mathrm{Aut}(X, \mu)$.

Pestov (1998) also produced another example: $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$. His proof however did not use concentration of measure techniques but rather finite combinatorics, more specifically the classical Ramsey Theorem. From this it also follows that $H_+([0,1])$ is extremely amenable.

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Metrizable universal minimal flows

The first example of calculation of a metrizable but non-trivial universal minimal flow is due to Pestov (1998): The universal minimal flow of $H_+(\mathbb{T})$ is $\mathbb{T}$. Two more examples were found later by Glasner-Weiss (2002,2003): The universal minimal flow of S_∞ is the space LO of linear orderings of $\mathbb{N}$. The universal minimal flow of $H(2^{\mathbb{N}})$ is the Uspenskii space of maximal chains of closed subsets of the Cantor space. These all used Ramsey techniques.

Universal minimal flows of automorphism groups

We will next discuss the study of extreme amenability and calculation of universal minimal flows for automorphism groups of countable structures. This was undertaken in a paper of K-Pestov-Todorcevic (2005). The main outcome of this paper is the development of a duality theory which shows that there is an equivalence between the structure of the universal minimal flow of the automorphism group of a Fraïssé structure and the Ramsey theory of its finite "approximations", i.e., its age.

Structural Ramsey theory

We first recall the classical Ramsey Theorem.

Given $n, m, k, M \geq 1$, with $k \leq m \leq M$, the notation

$$M \to (m)^k_n$$

means that if we color the k-element subsets of $\{1, \ldots, M\}$ with n colors, there is a subset X of $\{1, \ldots, M\}$ of size m which is monochromatic, i.e., all k-element subsets of X have the same color.

Theorem (Ramsey 1930)

For each $n, m, k \geq 1$, with $m \geq k$, there is $M \geq m$, such

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Structural Ramsey theory is a deep generalization of the classical Ramsey theorem to classes of finite structures. It was developed primarily in the 1970's by: Graham, Leeb, Rothschild, Nešetřil, Rödl, Prömel, Voigt, Abramson-Harrington, ...

Definition

A class $\mathcal{K}$ of finite structures (in the same signature) has the Ramsey Property if for any $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$, and any $n \geq 1$, there is $\boldsymbol{C} \geq \boldsymbol{B}$ in $\mathcal{K}$, such that

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_n.$$

Examples of classes with the Ramsey property:

- finite linear orderings (Ramsey)
- finite Boolean algebras (Graham-Rothschild)
- finite-dimensional vector spaces over a given finite field (Graham-Leeb-Rothschild)
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However, the class of finite graphs does not have the Ramsey property!

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Duality theory

We can now summarize in general terms the main point of the duality theory alluded to earlier:

Let $\mathcal{K}$ be a Fraïssé class of finite structures and $\boldsymbol{K} = \mathrm{Frlim}(\mathcal{K})$ its Fraïssé limit. Then we have a canonical correspondence:

structure of the UMF of $\mathrm{Aut}(\boldsymbol{K})$ $\leftrightarrow$ Ramsey theory of $\mathcal{K}$

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Extreme amenability of automorphism groups

We will first consider the problem of characterizing the extremely amenable automorphism groups. We have seen they are all of the form $G = \mathrm{Aut}(\boldsymbol{K})$ for a Fraïssé structure $\boldsymbol{K}$. But which automorphism groups of Fraïssé structures are extremely amenable?

Theorem (KPT)

Let $\mathcal{K}$ be a Fraïssé class and $\boldsymbol{K}$ its limit. Then the following are equivalent:

- $\mathrm{Aut}(\boldsymbol{K})$ *is extremely amenable.*
- $\mathcal{K}$ *consists of rigid structures and has the Ramsey property.*

Using the results of the structural Ramsey theory gives now a plethora of new examples of interesting extremely amenable groups.

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Extreme amenability of automorphism groups

Rigid Fraïssé class $\mathcal{K}$	Fraïssé limit $\boldsymbol{K}$
Ramsey property of $\mathcal{K}$	extreme amenability of $\mathrm{Aut}(\boldsymbol{K})$
linear orders	$\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$
ordered graphs	$\mathrm{Aut}(\langle \boldsymbol{R}, < \rangle)$
lex. ordered vector spaces	$\mathrm{Aut}(\langle \boldsymbol{V}_\infty, < \rangle)$
lex. ordered Boolean algebras	$\mathrm{Aut}(\langle \boldsymbol{B}_\infty, < \rangle)$
ordered rational metric spaces	$\mathrm{Aut}(\langle \boldsymbol{U}_\mathbb{Q}, < \rangle)$

Calculation of universal minimal flows

This duality theory also extends to the calculation of (non-trivial) metrizable universal minimal flows for automorphism groups. In certain situations one can assign to a Fraïssé class $\mathcal{K}$ with limit $\boldsymbol{K}$ a companion Fraïssé class $\mathcal{K}^*$ consisting of structures of the form $\langle \boldsymbol{A}, < \rangle$, obtained by adding to each structure $\boldsymbol{A}$ in $\mathcal{K}$ appropriate "admissible orderings". This gives rise to a canonical flow $X_{\mathcal{K}^*}$ of the automorphism group of $\boldsymbol{K}$. It is the compact, metrizable space of "admissible orderings" on $\boldsymbol{K}$, i.e., the linear orderings on $\boldsymbol{K}$ with the property that their restrictions to the finite substructures are admissible. Then we have the following:

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For each Fraïssé class $\mathcal{K}$ with limit $\boldsymbol{K}$, the following are equivalent:

- *$X_{\mathcal{K}^*}$ is the universal minimal flow of the automorphism group of $\boldsymbol{K}$.*
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The concept of the ordering property is also an important ingredient in the structural Ramsey theory that has been introduced by Nešetřil and Rödl in the 1970's.

Definition

For $\mathcal{K}, \mathcal{K}^*$ as above we say that $\mathcal{K}^*$ has the ordering property if for every $\boldsymbol{A}$ in $\mathcal{K}$ there is a $\boldsymbol{B}$ in $\mathcal{K}$ such that for any admissible ordering $<$ of $\boldsymbol{A}$ and any admissible ordering $<'$ of $\boldsymbol{B}$ there is an embedding of $\langle \boldsymbol{A}, < \rangle$ into $\langle \boldsymbol{B}, <' \rangle$.

Calculation of universal minimal flows

Examples

- $\mathcal{K}$ = finite graphs, $\boldsymbol{K} = \boldsymbol{R}$; $\mathcal{K}^*$ = finite ordered graphs. Then the UMF of $\mathrm{Aut}(\boldsymbol{R})$ is the space $X_{\mathcal{K}*}$ of all linear orderings of the random graph.
- $\mathcal{K}$ = finite sets, $\boldsymbol{K} = \langle \mathbb{N} \rangle$; $\mathcal{K}^*$ = finite orderings. Then the UMF of S_∞ is the space $X_{\mathcal{K}*}$ of all linear orderings on $\mathbb{N}$ (Glasner-Weiss).
- $\mathcal{K}$ = f.d. vector spaces over a fixed finite field, $\boldsymbol{K} = \boldsymbol{V}_\infty$; $\mathcal{K}^*$ = lex. ordered f.d. vector spaces. Then the UMF of the general linear group of $\boldsymbol{V}_\infty$ is the space $X_{\mathcal{K}^*}$ of all "lex. orderings" on $\boldsymbol{V}_\infty$.
- $\mathcal{K}$ = finite posets, $\boldsymbol{K} = \boldsymbol{P}$; $\mathcal{K}^*$ = finite posets with linear extensions. Then the UMF of $\mathrm{Aut}(\boldsymbol{P})$ is the space $X_{\mathcal{K}^*}$ of all linear extensions of the random poset.

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Applications of the duality theory

Duality establishes the equivalence between the structure of the universal minimal flow of a Fraïssé structure and the Ramsey properties of its age and therefore one can use the extensive structural Ramsey theory to analyze such universal minimal flows and discover many new examples of extremely amenable groups.

Also automorphism groups of Fraïssé structures often admit dense embeddings into other "larger" Polish groups. If G is extremely amenable and can be densely embedded in H, then H is also extremely amenable. Thus results concerning extreme amenability of automorphism groups, which use combinatorial methods, can be used to establish extreme amenability of other groups which were originally established by concentration of measure techniques.

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Part II. Ergodic theory

Amenable groups

I will next discuss some very recent work of Omer Angel, K. and Russ Lyons in the ergodic theory of automorphism groups of Fraïssé structures.

Let G be a Polish group acting continuously on a compact space X, i.e., X is a G-flow. We will be looking at invariant Borel probability measures for such an action. In general such measures might not exist.

Definition

The group G is called amenable if every G-flow admits an invariant Borel probability measure.

In particular every extremely amenable group is amenable. On the other hand S_∞ is amenable but not extremely amenable and the automorphism group of the countable atomless Boolean algebra is not amenable.

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A particularly important class of automorphism groups that turn out to be amenable is the following.

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Let $\mathcal{K}$ be a Fraïssé class of finite structures. We say that $\mathcal{K}$ is a Hrushovski class if for any $\boldsymbol{A}$ in $\mathcal{K}$ there is $\boldsymbol{B}$ in $\mathcal{K}$ containing $\boldsymbol{A}$ such that any partial automorphism of $\boldsymbol{A}$ extends to an automorphism of $\boldsymbol{B}$.

Some basic examples of such classes are the pure sets, graphs (Hrushovski), rational valued metric spaces (Solecki), finite dimensional vector spaces over finite fields, etc.

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Let $\mathcal{K}$ be a Fraïssé class of finite structures and $\boldsymbol{K}$ its Fraïssé limit. If $\mathcal{K}$ is a Hrushovski class, then we say that $\boldsymbol{K}$ is a Hrushovski structure.

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Let $\mathcal{K}$ be a Fraïssé class of finite structures and $\boldsymbol{K}$ its Fraïssé limit. If $\mathcal{K}$ is a Hrushovski class, then we say that $\boldsymbol{K}$ is a Hrushovski structure.

Hrushovski structures

This turns out to be a property of automorphism groups:

Proposition (K-Rosendal)

Let $\mathcal{K}$ be a Fraïssé class of finite structures and $\boldsymbol{K}$ its Fraïssé limit. Then the following are equivalent

- $\boldsymbol{K}$ *is a Hrushovski structure.*
- $\mathrm{Aut}(\boldsymbol{K})$ *is compactly approximable, i.e., there is a increasing sequence K_n of compact subgroups whose union is dense in the automorphism group.*

In particular the automorphism group of a Hrushovski structure is amenable.

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Unique ergodicity

Definition

If G is an amenable group and X a G-flow, then we say that this flow is uniquely ergodic if there is a unique invariant probability Borel measure (which then must be ergodic).

We now consider the following property of groups.

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Unique ergodicity

Trivially every extremely amenable group and every compact group is uniquely ergodic. Glasner and Weiss have shown that S_∞ is uniquely ergodic. On the other hand, Weiss has shown that no infinite countable group is uniquely ergodic and he believes that this extends to non-compact locally compact groups, although this has not been checked in detail yet.

Unique ergodicity as a quantitative version of the Ordering Property

Interestingly it turned out that unique ergodicity fits well in the framework of the duality theory of KPT (which originally was developed in the context of topological dynamics). In many cases it can simply be viewed as a quantitative version of the Ordering Property.

Definition (AKL)

Let $\mathcal{K}^*$ be a companion of $\mathcal{K}$. We say that $\mathcal{K}^*$ satisfies the Quantitative Ordering Property (QOP) if the following holds:

There is an isomorphism invariant map that assigns to each structure $\boldsymbol{A}^* = \langle \boldsymbol{A}, < \rangle \in \mathcal{K}^*$ a real number $\rho(\boldsymbol{A}^*)$ in $[0,1]$ such that for every $\boldsymbol{A} \in \mathcal{K}$ and each $\epsilon > 0$, there is a $\boldsymbol{B} \in \mathcal{K}$ and a nonempty set of embeddings $E(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{A}$ into $\boldsymbol{B}$ with the property that for each $\mathcal{K}^*$-admissible ordering $<$ of $\boldsymbol{A}$ and each $\mathcal{K}^*$-admissible ordering $<'$ of $\boldsymbol{B}$ the proportion of embeddings in $E(\boldsymbol{A}, \boldsymbol{B})$ that preserve $<, <'$ is equal to $\rho(\langle \boldsymbol{A}, < \rangle)$, within ϵ.

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Unique ergodicity as a quantitative version of the Ordering Property

For example, if $\mathcal{K}$ is the class of finite graphs, where $\mathcal{K}^*$ is the class of ordered finite graphs, one can establish QOP by showing that for any finite graph $\boldsymbol{A}$ with n vertices and $\epsilon > 0$, there is a graph $\boldsymbol{B}$, containing a copy of $\boldsymbol{A}$, such that given any orderings $<$ on $\boldsymbol{A}$ and $<'$ on $\boldsymbol{B}$, the proportion of all embeddings of $\boldsymbol{A}$ into $\boldsymbol{B}$ that preserve the orderings $<, <'$ is, up to ϵ, equal to $1/n!$.

Unique ergodicity as a quantitative version of the Ordering Property

Theorem (AKL)

Let $\mathcal{K}^*$ be a companion of $\mathcal{K}$ and let G be the automorphism group of the Fraïssé limit of $\mathcal{K}$ and assume that G is amenable. Then QOP implies the unique ergodicity of G. Moreover, if $\mathcal{K}$ is a Hrushovski class, QOP is equivalent to the unique ergodicity of G.

Proving unique ergodicity

By more direct means (but still using the calculation of the UMF), one can show that the following automorphism groups are uniquely ergodic:

- S_∞ (Glasner-Weiss)
- The isometry group of the Baire space and various ultrametric Urysohn spaces (AKL)
- The general linear group of the (countably) infinite-dimensional vector space over a finite field (AKL)

By applying now the preceding QOP criterion and probabilistic arguments (deviation inequalities) one can now also show the following:

Theorem (AKL)

The automorphism groups of the following structures are uniquely ergodic

- *The random graph*
- *The random K_n-free graph*
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Unique Ergodicity Problem

In fact I do not know any counterexample to the following problem:

Problem (Unique Ergodicity Problem)

Let G be a non-Archimedean group with a metrizable universal minimal flow. If G is amenable, then is it uniquely ergodic?

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Part III. Unitary representations

Unitary representations

Definition

Let G be a Polish group. A continuous representation of G is a continuous action of G on a (complex) Hilbert space H by unitary transformations. It is irreducible if it has no non-trivial closed (linear) subspaces.

A goal of representation theory is to describe the irreducible representations and understand how other representations are build out of the irreducible ones.

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A goal of representation theory is to describe the irreducible representations and understand how other representations are build out of the irreducible ones.

The Peter-Weyl Theorem

A classical example of such an analysis is the Peter-Weyl Theorem for compact groups.

Theorem (Peter-Weyl)

Let G be a compact metrizable group.

- *There are only countably many irreducible representations of G and every representation of G is a direct sum of irreducible representations.*
- *The irreducible representations are all finite dimensional and the left-regular representation of G is the direct sum of all the irreducible representations each appearing with a multiplicity equal to its dimension.*

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An analog of Peter-Weyl for oligomorphic groups

Recently Tsankov proved a very interesting analog of Peter-Weyl for oligomorphic automorphism groups, which in general are far from compact. This is a major extension of earlier results that were proved, by different methods, for S_∞ itself and (a variant of) the general linear group of the (countable) infinite dimensional vector space over a finite field by Lieberman and Olshanski.

Definition

An automorphism group is oligomorphic if for each $n \geq 1$ its action on n-tuples (of the underlying structure) has only finitely many orbits. Equivalently, by a theorem of Engeler, Ryll-Nardzewski and Svenonius, these are the automorphism groups of $\aleph_0$-categorical (Fraïssé) structures.

For example the automorphism group of any relational Fraïssé structure in a finite signature has this property.

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Moreover in many cases Tsankov provides an explicit description of the irreducible representations. For example, for the automorphism group of the random graph one obtains all the irreducible representations by "lifting" through the process of induction the irreducible representations of the automorphism groups of finite graphs. Also for the automorphism group of the rational order, the irreducible representations are exactly the actions of this group on $\ell^2([\mathbb{Q}]^n)$, where $[\mathbb{Q}]^n$ is the set of n elements subsets of $\mathbb{Q}$.

Property (T)

Finally Tsankov uses his analysis to show that many oligomorphic groups have Kazhdan's propery (T).

Definition

A topological group G has property (T) if there is compact $Q \subseteq G$ and $\epsilon > 0$ such every unitary representation of G that has a unit (Q, ϵ)-invariant vector actually has a unit invariant vector.

Such groups include S_∞ and the automorphism groups of the random graph, the atomless countable Boolean algebra, the rational order and the countable infinite-dimensional vector space over a finite field (with actually a Q of size 2.).

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