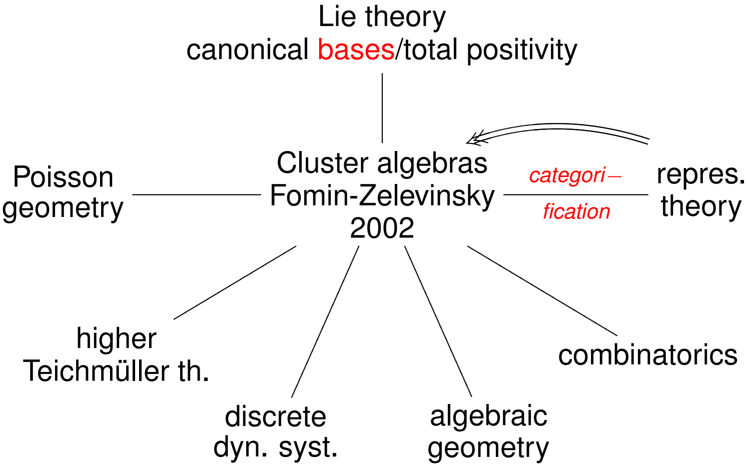
Cluster algebras and cluster monomials

Bernhard Keller

Université Paris Diderot – Paris 7

6 ECM Kraków, 5 lipca 2012 roku

Context


Lie theory
canonical bases/total positivity
Cluster algebras
Fomin-Zelevinsky
2002
Poisson geometry
categori–
fication
repres. theory
higher Teichmüller th.
discrete dyn. syst.
algebraic geometry
combinatorics

Hope, Conjecture, Theorem

Hope (Fomin-Zelevinsky)

Each cluster algebra admits a ‘canonical basis’ containing the cluster monomials.

Conjecture (FZ, 2002)

The cluster monomials are linearly independent.

Theorem (Cerulli–Labardini–K–Plamondon, 2012)

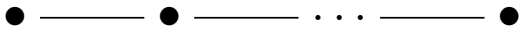
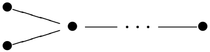
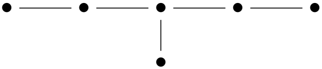
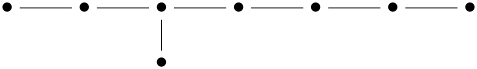
The conjecture holds for all cluster algebras associated with quivers.

Some previous partial results

Caldero–K (2008), Fomin–Shapiro–Thurston (2008), Derksen–Weyman–Zelevinsky (2010), Demonet (2011), Plamondon (2011), Geiss–Leclerc–Schröer (2012).

Plan

The Dynkin diagrams (of type ADE)

Name	Graph	n	# pos. roots
A_n		≥ 1	$n+1$
D_n		≥ 4	$2n-2$
E_6		6	36
E_7		7	63
E_8		8	120

A quiver is an oriented graph

Definition

A quiver Q is an oriented graph: It is given by

- a set Q_0 (the set of vertices)
- a set Q_1 (the set of arrows)
- two maps
  - $s : Q_1 \to Q_0$ (taking an arrow to its source)
  - $t : Q_1 \to Q_0$ (taking an arrow to its target).

Remark

A quiver is a 'category without composition'.

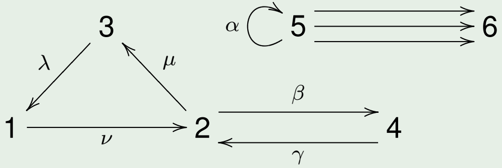
A quiver can have loops, cycles, several components.

Example

The quiver $\vec{A}_3$: $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ is an orientation of the Dynkin diagram A_3 : $1 \text{ --- } 2 \text{ --- } 3$.

Example

Q :



We have $Q_0 = \{1, 2, 3, 4, 5, 6\}$, $Q_1 = \{\alpha, \beta, \ldots\}$.
α is a loop, (β, γ) is a 2-cycle, (λ, μ, ν) is a 3-cycle.

Definition of quiver mutation

Let Q be a finite quiver without loops nor 2-cycles (from now on always assumed).

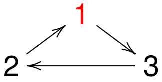
Definition (Fomin-Zelevinsky)

Let $j \in Q_0$. The mutation $\mu_j(Q)$ is the quiver obtained from Q as follows

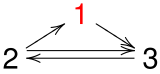
1) for each subquiver $i \xrightarrow{\beta} j \xrightarrow{\alpha} k$, add a new arrow
$$i \xrightarrow{[\alpha\beta]} k \,;$$
2) reverse all arrows incident with j;
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles (e.g. $\bullet \rightleftarrows \!\!\! \rightarrow \bullet$ yields $\bullet \longrightarrow \bullet$, '2-reduction').

Examples of quiver mutation

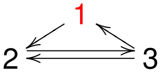
A simple example:



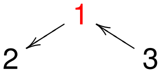
1)



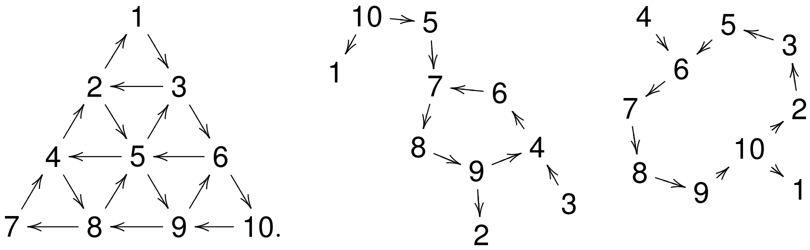
2)



3)



More complicated examples: Google ‘quiver mutation’!



Recall: We wanted to define cluster algebras!

Seeds and their mutations

Definition

A seed is a pair (R, u), where

a) R is a quiver with vertices $1, \ldots, n$.

b) $u = \{u_1, \ldots, u_n\}$ is a free generating set of the field $\mathbb{Q}(x_1, \ldots, x_n)$.

Example: $(1 \to 2 \to 3, \{x_1, x_2, x_3\}) = (x_1 \to x_2 \to x_3)$.

Definition

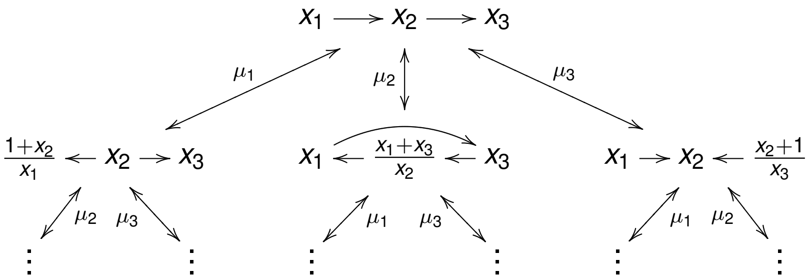
For a vertex j of R, the mutation $\mu_j(R, u)$ is (R', u'), where

a) $R' = \mu_j(R)$;

b) $u' = u \setminus \{u_j\} \cup \{u'_j\}$, with u'_j defined by the exchange relation

$$u_j u'_j = \prod_{\substack{\text{arrows} \\ i \to j}} u_i + \prod_{\substack{\text{arrows} \\ j \to k}} u_k .$$

An example


$x_1 \longrightarrow x_2 \longrightarrow x_3$
μ_1
μ_2
μ_3
$\frac{1+x_2}{x_1} \longleftarrow x_2 \longrightarrow x_3$
$x_1 \longleftarrow \frac{x_1+x_3}{x_2} \longleftarrow x_3$
$x_1 \longrightarrow x_2 \longleftarrow \frac{x_2+1}{x_3}$
μ_2 μ_3
μ_1 μ_3
μ_1 μ_2

Clusters, cluster variables and the cluster algebra

Let Q be a quiver with n vertices.

Definition

a) The initial seed is $(Q, x) = (Q, \{x_1, \ldots, x_n\})$.

b) A cluster is an n-tuple u appearing in a seed (R, u) obtained from (Q, x) by iterated mutation.

c) The cluster variables are the elements of the clusters.

d) The cluster algebra $\mathcal{A}_Q$ is the subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by the cluster variables.

e) A cluster monomial is a product of powers of cluster variables which all belong to the same cluster.

Remark

If Q is mutation equivalent to Q', then $\mathcal{A}_Q \xrightarrow{\sim} \mathcal{A}_{Q'}$.

Sergey Fomin
University of Michigan

Andrei Zelevinsky
Northeastern University

Fundamental properties

Let Q be a connected quiver.

Theorem (Fomin-Zelevinsky, 2002)

a) *All cluster variables are Laurent polynomials.*
b) *There is only a finite number of cluster variables iff Q is mutation-equivalent to an orientation $\vec{\Delta}$ of a Dynkin diagram Δ. Then Δ is unique and called the cluster type of Q.*
c) *If $Q = \vec{\Delta}$, the non initial cluster variables are in canonical bijection with the positive roots of the associated root system.*

Example: D_4.

Finite case

Theorem (Caldero-K, 2008)

The cluster monomials form a basis of the cluster algebra $\mathcal{A}_Q$ iff there is only a finite number of cluster variables.

Example: $Q = \vec{A}_2 = (1 \to 2)$

- There are 5 cluster variables

$$x_1\ ,\ x_2\ ,\ x_1' = \frac{1+x_2}{x_1}\ ,\ x_2' = \frac{x_1+1+x_2}{x_1 x_2}\ ,\ x_1'' = \frac{1+x_1}{x_2}.$$

- The clusters $u = \{u_1, u_2\}$ are the 5 pairs of consecutive cluster variables in the cyclic order.
- There are 5 families of cluster monomials $u_1^{m_1} u_2^{m_2}$, $m_1, m_2 \geq 0$. Their union is a basis.

Linear independence: general case

Theorem (Cerulli–Labardini–K–Plamondon, 2012)

The cluster monomials of any cluster algebra associated with a quiver are linearly independent.

Example: $Q: \; 1 \rightrightarrows 2$ (Kronecker quiver)

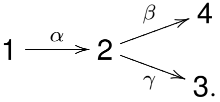
- There are countably many cluster variables

$$\ldots\,,\; v_{-1}\,,\; v_0\,,\; v_1 = x_1\,,\; v_2 = x_2\,,\; v_3\,,\; v_4\,,\; \ldots$$

- The clusters $u = \{u_1, u_2\}$ are all pairs of consecutive cluster variables $\{v_i, v_{i+1}\}$.
- There are countably many families of cluster monomials $u_1^{m_1} u_2^{m_2}$, $m_1, m_2 \geq 0$. Their union is linearly independent.

representation of a quiver = diagram of vector spaces

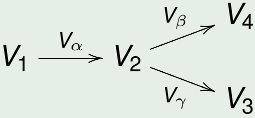
Let Q be a quiver with vertices $1, \ldots, n$, e.g.



Definition

A representation of Q is a diagram of finite-dimensional complex vector spaces of the shape given by Q.

Example



The quiver Grassmannian

Let V be a representation of Q.

Definition

- A subrepresentation $V' \subset V$ is a family of subspaces $V'_i \subset V_i$ such that, for each arrow $\alpha : i \to j$, we have $V_\alpha(V'_i) \subset V'_j$.
- Let $e \in \mathbb{N}^n$. The quiver Grassmannian is the variety $Gr_e(V)$ of all subrepresentations $V' \subset V$ such that $\dim V'_i = e_i$ for all i.

Remarks

- $Gr_e(V)$ is a closed subvariety in $\prod_{i=1}^n Gr_{e_i}(V_i)$.
- Every projective variety is a quiver Grassmannian (Reineke, 2012).

Cluster monomials from quiver representations

Theorem (Caldero–Chapoton, ..., Derksen-Weyman-Z. 2010)

For each cluster monomial M of $\mathcal{A}_Q$, there is a representation V of Q such that

$$M = m_V \sum_{e \in \mathbb{N}^n} \chi(Gr_e(V)) \prod_{j=1}^{n} (\prod_{i=1}^{n} x_i^{b_{ji}})^{e_j}$$

where

- m_V *is an explicit monomial in* $x_1, \ldots, x_n$,
- $\chi =$ *Euler characteristic,*
- $(b_{ij}) = B_Q$ *is the skew-symmetric matrix with*

$$b_{ij} = \#(\text{arrows } i \to j \text{ in } Q) - \#(\text{arrows } j \to i \text{ in } Q).$$

On the proof of the independence theorem

The proof of the independence theorem uses

a) the above explicit expressions

$$M = m_V \sum_{e \in \mathbb{N}^n} \chi(Gr_e(V)) \prod_{j=1}^{n} (\prod_{i=1}^{n} x_i^{b_{ji}})^{e_j}$$

b) precise information on V encoded in the cluster category (Plamondon, Ph. D. thesis, 2011).

Open problems

- Generalize the independence theorem from quivers to valued quivers (which correspond to skew-symmetrizable matrices B_Q). Partial results: …, Demonet 2011
- Show that the coefficients of all cluster monomials are non negative integers (FZ's positivity conjecture).
  Partial results:
  - …
  - Hernandez-Leclerc, Nakajima, Qin, Kimura–Qin (05/2012): Q mutation-equivalent to a quiver without oriented cycles.
  - …

Summary

- Cluster algebras are commutative algebras with a rich combinatorial structure.
- They appear in a great variety of subjects ranging from higher Teichmüller theory to discrete dynamical systems.
- They contain the cluster monomials, which are linearly independent but usually do not form a basis.
- Do not forget to google quiver mutation!