The complex Monge-Ampère equation

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- The Dirichlet problem for the complex Monge-Ampère operator
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- The complex Monge-Ampère equation on compact Kähler manifolds
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Definitions

- A function u defined in a domain Ω of $\mathbb{C}^n$, taking values in $[-\infty, +\infty)$, and not identically $-\infty$, is called plurisubharmonic (shortly psh) if it is upper semicontinuous in Ω and subharmonic on any intersection of a complex line with Ω .

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- When a psh function is of class C^2 then the complex Hessian $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})$ is positive semidefinite, that is

$$\sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0, \quad w \in \mathbb{C}^n.$$

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- For $d = (\partial + \bar{\partial})$ and $d^c = i\,(\bar{\partial} - \partial)$ the above inequality says that the differential form $dd^c u = 2i\partial\bar{\partial} u$ is positive.

Definitions

- For a collection of smooth psh functions $u_1, u_2, \ldots, u_k$ the form

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k$$

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$$cap(E, \Omega) = \sup\{\int_E (dd^c u)^n : u \in PSH(\Omega), -1 \leq u < 0\}.$$

- It vanishes exactly on pluripolar sets ($-\infty$ sets of psh functions).

The Dirichlet problem

- The complex Monge-Ampère operator

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = const\,(dd^c u)^n \quad \text{(wedge product).}$$

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- The Dirichlet problem in a strictly pseudoconvex domain

$$u \in PSH \cap L^\infty(\Omega)$$
$$(dd^c u)^n = f\, dV \quad (f \geq 0, \quad dV \text{ Lebesgue measure})$$
$$\lim_{\zeta \to z} u(\zeta) = \varphi(z) \quad z \in \partial\Omega, \; \varphi \in C(\partial\Omega)$$

Later $f\,dV$ will be replaced by a Borel measure $d\mu$.

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- Caffarelli, Kohn, Nirenberg, Spruck (1985) proved regularity of solutions for smooth data and $f > 0$. Their approach followed the one taken for the real Monge-Ampère equation (both are fully nonlinear elliptic PDEs).

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- create a dictionary between classes of solutions (continuous ones, bounded, etc.) and corresponding families of measures on the right hand side;
- determine the largest class of psh functions for which the Monge-Ampère operator is well defined.

THE DIRICHLET PROBLEM

- Regularity of the solution (assuming "nice" boundary data) depends on a bound

$$\mu(E) \leq F(cap(E, \Omega)), \tag{0.1}$$

for a continuous real function $F \geq 0$ which vanishes at the origin.

The Dirichlet problem

Basic lemma, K. '96

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Assume $v \in PSH \cap C(\Omega)$, and $u \in PSH \cap L^\infty(\Omega)$. Suppose that (0.1) holds for $\mu = (dd^c u)^n$, any compact set E and

$$F(t) = ct \log^{-m}(t^{-1/n}), \quad c > 0, \; m > n; \quad (\text{or} \quad F(t) = ct^{1+\epsilon}, \; \epsilon > 0)$$

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If the sublevel sets $U(s) := \{u - s < v\}$ are nonempty and relatively compact in Ω for s in the range $[s_0, s_0 + \alpha]$ then the number α is bounded by

$$\kappa(cap(U(s_0 + \alpha), \Omega))$$

for a function κ vanishing at the origin and explicitly given in terms of F .

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- How does this lemma yield L^∞ estimates?
- Fix $v = 0$ and $\alpha > 0$. Since, by Chern-Levine-Nirenberg inequalities it is known that capacity of sublevel sets $\{u < -t\}$ is bounded by $const/t$, the lemma gives a lower bound for s_0.

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Suppose $u \in PSH(\Omega) \cap C(\overline{\Omega}), u = 0$ on $\partial\Omega$, $\int (dd^c u)^n \leq 1$. Then for any $a < 2$ the Lebesgue measure $V(\Omega_s)$ of the set $\Omega_s := \{u < s\}$ is bounded from above by $c \exp(-2\pi a|s|)$, where c does not depend on u.

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- More precise estimates of this kind, by Demailly and Åhag-Cegrell-K.-Phạm-Zeriahi provide an alternative proof of an inequality from local algebra due to de Fernex, Ein, Mustaţă

$$lc(I) \geq ne(I)^{-1/n},$$

where I is an ideal of germs of holomorphic functions with isolated singularity at the origin in $\mathbb{C}^n$, $lc(I)$ denotes the log canonical threshold of I and $e(I)$ - the Hilbert-Samuel multiplicity of the ideal.

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- The theorem itself remains true for f belonging to some broader Orlicz spaces and this general result is close to being optimal.
- Problem: The precise characterization of measures yielding bounded (resp. continuous solutions) is not known.

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- Problem: Is it true for continuous solutions?
- The M-A operator is also well defined for some unbounded psh functions.
- For functions with too strong singularities, like $u(z) = \log |z_j|$ a reasonable definition is not possible.

The Dirichlet problem

- Cegrell '98 initiated the study of classes of psh functions of "finite energy". Using a Hölder like inequality

$$\int_\Omega (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j}$$
$$\leq C_{j,p} \Big(\int_\Omega (-u)^p (dd^c u)^n\Big)^{\frac{p+j}{n+p}} \Big(\int_\Omega (-v)^p (dd^c v)^n\Big)^{\frac{n-j}{n+p}}$$

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  he considered families of functions having finite integrals on the right hand side of the inequality.

Theorem, Cegrell '04

The Dirichlet problem has a unique solution for any measure of finite total mass which vanishes on pluripolar sets (with relaxed boundary condition).

The Dirichlet problem

- Cegrell defined a cone $\mathcal{E}$ in the set of psh functions in a given open set which turned out to be the maximal domain of definition of the M-A operator.

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- This domain was explicitly characterized by Błocki '06. In particular, in dimension 2, it contains exactly psh functions in the Sobolev space $W^{1,2}$. It is defined by local conditions.
- Problem: Which measures carried by pluripolar sets are M-A measures of psh functions?

- Consider a compact n-dimensional Kähler manifold M with the fundamental form ω given in local coordinates by

$$\omega = \frac{i}{2} \sum_{k,j} g_{k\bar{j}} dz^k \wedge d\bar{z}^j.$$

The matrix $(g_{k\bar{j}})$ is positive definite and Hermitian symmetric.

M-A equation on compact Kähler manifolds

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The matrix $(g_{k\bar{j}})$ is positive definite and Hermitian symmetric.

- The M-A equation takes now the form

$$(\omega + dd^c u)^n = f\,\omega^n, \quad \omega + dd^c u \geq 0, \tag{0.2}$$

(u satisfying the inequality is called ω-psh). The given function $0 \leq f \in L^1(M)$ is normalized by the condition $\int_M f\,\omega^n = \int_M \omega^n$.

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- Calabi: solution gives a new Kähler metric with preassigned Ricci curvature

$$Ric(\omega) = -\frac{1}{2} dd^c [\log \det(g_{j\bar{k}})].$$

- Calabi conjectured that given a $(1,1)$ closed form R' on (M, ω) representing the first Chern class one can find a Kähler metric ω' (in given Kähler class) with $Ric(\omega') = R'$.

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Theorem, Yau '78

Let $f > 0, f \in C^k(M), k \geq 3$. Then there exists a unique solution of (0.2) satisfying $\max u = 0$ and belonging to Hölder class $C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$.

M-A equation on compact Kähler manifolds

- To find Kähler-Einstein metrics $Ric(\omega) = const\,\omega$ one needs to solve the equation of M-A type

$$(\omega + dd^c u)^n = \exp(cu + f)\omega^n,$$

where the constant c depends on the first Chern class $c_1(M)$.

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Theorem, Aubin '78, Yau '78

If f satisfies a normalizing condition, is positive, smooth and its partial derivative with respect to the first variable is strictly positive then the equation

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- This gives existence of K-E metrics for $c_1(M) < 0$.

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If $f \in L^p(\omega^n)$, $p > 1$ is suitably normalized then (0.2) has a (unique up to additive constant) continuous solution.

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If $f \in L^p(\omega^n)$, $p > 1$ is suitably normalized then (0.2) has a (unique up to additive constant) continuous solution.

- The solutions are Hölder continuous (K '08) and the Hölder exponent is arbitrarily close to $2/(1 + nq)$ (q the conjugate to p) (Demailly et all '11). It cannot be better than $2/nq$.

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Theorem, K '03, Dinew, Zhang '10

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Suitably normalized ω-psh solutions of the equations

$$(dd^c u + \omega)^n = f\omega^n, \quad (dd^c v + \omega)^n = g\omega^n,$$

for $f, g \in L^p, \quad p > 1$ satisfy, given any $\epsilon > 0$,

$$||u - v||_\infty \leq c||f - g||_{L^1(\omega^n)}^{1/(n+\epsilon)},$$

with c depending only on p, ϵ and L^p norms of f and g.

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- Stability is used to show Hölder continuity above and other regularity statements.

M-A equation on compact Kähler manifolds

- Problem On compact manifolds the standard smoothing technique, via the convolution, causes a loss of positivity of $\omega + dd^c u$. To deal with the curvature a "geodesic" regularization of Demailly coupled with his method of attenuation of singularities (via Kiselman's minimum principle), is employed. Are there any other clever ways of smoothing, especially in the presence of singularities ?

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- They defined $(\omega + dd^c u)^n$ on the set $M \setminus \{u = -\infty\}$ and took it as the definition of the M-A measure $(\omega + dd^c u)^n$ on M provided that

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- Such a measure may not be locally well defined !

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- Dinew '09 proved uniqueness of those solutions. His new method works also in the setting of big cohomology classes.
- For Dirac measures the uniqueness no longer holds.
- To prove the above theorem for big rather than Kähler classes Berman, Boucksom, Guedj and Zeriahi employed a variational method for the Aubin-Mabuchi energy functional

$$E(u) = \frac{1}{n+1} \sum_k \int_M u(\omega + dd^c u)^k \wedge \omega^{n-k}.$$

This approach gives weak solutions without referring to the Calabi-Yau theorem.

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- Cao '85: if $c_1(M)$ is negative or zero then for any initial metric the Kähler-Ricci flow converges to the unique K-E metric.
- Perelman (resp. Tian, Zhu '07): the same is true for $c_1(M) > 0$ provided that a K-E metric (resp. Kähler-Ricci soliton) exists.

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- Song, Tian '07: Minimal Model Program with Ricci Flow - any projective variety can be deformed by Kähler-Ricci flow with surgeries, to its minimal (and then canonical) model equipped with a singular K-E metric. This singular setting requires L^∞ estimates for weak solutions and stability results.

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- Similar applications in the study of metrics with cone singularities.

M-A equation on compact Hermitian manifolds

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- Problem Prove or disprove the existence of solutions for $f \in L^p,\ p > 1$, on general compact Hermitian manifolds.

Complex Hessian equations

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$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n$$

(ω a Kähler form, $f \geq 0$ the given function, $1 < m < n$) interpolate between the Laplace and the complex Monge-Ampère equation. The solutions should satisfy

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- Dinew, K. '12: there exist weak continuous solutions of the equation for $f \in L^p(\omega^n)$, $p > n/m$, and the bound on p is sharp.

THANK YOU FOR YOUR ATTENTION