Reinforced random walk

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Abstract. We show that linearly reinforced random walk has a recurrent phase in every graph. The proof does not use the magic formula.

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1. Introduction

Let f be some function from the positive integers $\{0, 1, 2, \dots\}$ into $(0, \infty)$, and let G be a graph and x_0 some vertex of it. Then we define *reinforced random walk* on G starting at x_0 with reinforcement function f as follows. It is a walk on G i.e. a random sequence of vertices $x_0, x_1, \dots$ with x_{i+1} a neighbour of x_i; and the transition probabilities depend only on the past. Specifically, for an edge e and time t we define the number of traversals $N(e, t)$ using

$$N(e,t) = \#\{s \text{ such that } 1 \le s \le t,\ (x_{s-1}, x_s) = e\}$$

where (x_{s-1}, x_s) is an unordered edge, so N counts traversals of e in both directions. Now apply f and normalize to get probabilities. Totally, the definition is

$$\mathbb{P}(x_{t+1} = x \,|\, x_0, \dots, x_t) = \frac{f(N((x, x_t), t))}{\sum\limits_{y \sim x_t} f(N((y, x_t), t)}$$

where $\sim$ is the neighbourhood relation in our graph. This finishes the definition of the process. This is well defined assuming all degrees of all vertices are finite, and we assume this from now on on all our graphs. There are also versions where reinforcement is applied to vertices or to directed edges, but for simplicity we concentrate for now on edge reinforced walks.

It comes as no surprise that different f give different processes, sometimes dramatically so. To make things more concrete let us discuss a few examples.

(1) Once-reinforced random walk is the process defined by

$$f(n) = \begin{cases} 1 & n = 0 \\ 2 & n \ge 1 \end{cases}.$$

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In other words, edges which have already been traversed are given double weight when compared to unknown edges, but the number of traversals does not matter.

(2) Linearly reinforced random walk is the process defined by $f(n) = 1 + n$. In other words, whenever you traverse an edge, you increase its weight by 1.

(3) Strongly reinforced random walk, where f grows faster than linear. For example, one may take $f(n) = 1 + n^2$.

Perhaps surprisingly, the order of difficulty seems to be decreasing. Once-reinforced random walk is the least understood, while the superlinear case is the simplest.

Let us start with the superlinear case. It turns out that in this case the walk gets stuck on a single edge, going back and forth endlessly. To understand why, let us prove the following weaker claim:

Theorem 1. *Let G be an arbitrary graph, x_0 a vertex of G and e an edge going out of x. Examine reinforced random walk on G starting from x_0 with reinforcement function $1+n^2$. Then with positive probability the process goes back and forth over e endlessly i.e. for all t, $x_{2t} = x_0$ and x_{2t+1} is the other vertex of e.*

Proof. Denote the other vertex of e by y. Let E_s be the event that $x_{2t} = x_0$ and $x_{2t+1} = y$ for all $2t \le s$ and $2t+1 \le s$, respectively. Let us calculate the probability of E_{s+1} given E_s. The numbers of traversals are $N(e, s) = s$ and $N(f, s) = 0$ for any edge $f \ne e$. Hence the probability is

$$\frac{f(s)}{f(s) + (\deg x_s - 1)f(0)} = \frac{1 + s^2}{1 + s^2 + (\deg x_s - 1)} \ge 1 - \frac{C}{s^2}.$$

for some constant C that depends only on the degrees of x_0 and y. Therefore the probability of E_s is

$$\mathbb{P}(E_s) = \prod_{u=1}^{s} \mathbb{P}(E_u | E_{u-1}) \ge \prod_{u=1}^{s} \left(1 - O\left(\frac{1}{u^2}\right)\right) \ge c$$

for some positive c. This proves the claim. □

There is no zero-one law for the behaviour of processes like reinforced random walk, so one cannot easily conclude from the theorem above that some event of interest happens with probability 1. Nevertheless, this has been proved with almost complete generality.

Theorem 2. *Let G be a graph and f a function satisfying*

$$\sum_{n=0}^{\infty} \frac{1}{f(n)} < \infty.$$

Then reinforced random walk on G with reinforcement function f eventually gets stuck on a single edge with probability 1, under either of the following conditions

(1) *f is increasing, or*

(2) *G contains no cycle of odd order.*

See Limic [15] and Limic and Tarrès [16, 17]. It is conjectured that neither condition is necessary, but clearly, the result is quite complete as is.

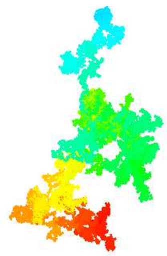
2. Phase transitions

Let us now consider once-reinforced random walk, but let us add a parameter a and examine the reinforcement function

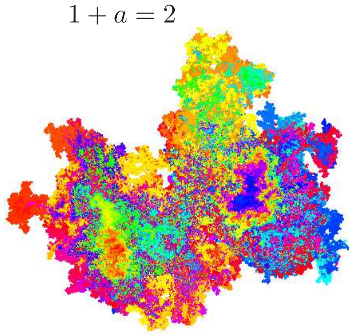
$$f(n) = \begin{cases} 1 & n = 0 \\ 1 + a & n \geq 1. \end{cases}$$

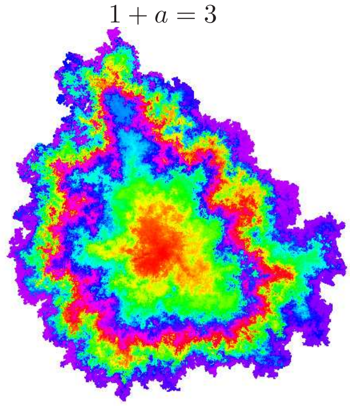
An exciting conjecture due to Sidoravicius is that, say for the graph $\mathbb{Z}^d$, $d \geq 2$, there is a *phase transition* in a. For a very small the walk is essentially simple random walk. For a very large the walk looks like a man trapped in a balloon: it constantly jumps against the boundaries of the balloon, occasionally inflating it, but the balloon keeps its round shape, as it inflates. The "balloon", of course, is the set of visited sites. In particular for $d \geq 3$ there should be a phase transition between recurrence and transience. Let us stress that a does not depend on t. It is not difficult to establish results if one first fixes t and then makes a very small or very large. In this case for $a \ll \frac{1}{t}$ one can directly check that the process is identical to simple random walk (with high probability), while for $a \gg t$ the process gets stuck on its first edge. But the conjecture is that for a fixed a there is a marked difference in behaviour as $t \to \infty$. Some simulation results can be seen in figure 1. Each simulation was done on a 5000×5000 grid and stopped when the process reached the boundaries. Colour depicts the number of reinforced edge. This means that the second edge which is reinforced is given colour 2, not the second edge which is traversed (colour change rate is not identical in the 6 images). The fractal-like colour patterns seen in $1 + a = 4$ are indicative of the process leaving holes all around and then going back to fill them at a macroscopically later time. Presumably 4 is quite close to the point of criticality where the processes of roughening and smoothing balance exactly.

The case of large a in this conjecture is related to a series of similar models. Internal diffusion-limited aggregation is a process where the walker, upon reaching a new vertex, is transported to the starting point x_0. For this model it is known since the early 90s that the set of visited vertices is spherical [14, 13], with impressive improvements in the precision of the results achieved lately [11, 12, 2]. A variation where, when the walker, upon reaching a new vertex is transported to a random vertex in the existing cluster also gives a ball [3]. Getting closer to once-reinforced random walk we may examine *excited to the center*. In this model the walker, upon getting to a new vertex gets a small drift towards the center. Less is known about this problem, basically only that the process is recurrent in any dimension,

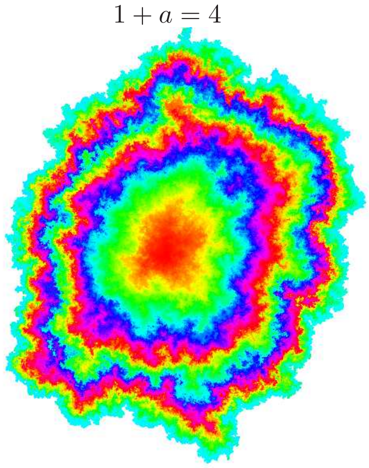


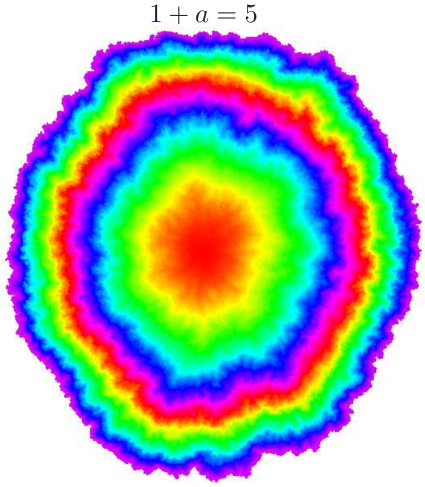
$1 + a = 2$ $1 + a = 3$





$1 + a = 4$ $1 + a = 5$





$1 + a = 6$ $1 + a = 10$

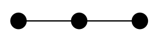
Figure 1. Once-reinforced random walk

regardless of the strength of the drift [unpublished, 2006]. Once-reinforced has all the difficulties of excited to the center and more: there is no preferred direction and there is a phase transition. Hence it seems quite challenging at this point. Existing rigorous results on the model are on a tree [7], and it is not difficult to solve the one-dimensional case, but neither case exhibits a phase transition, so for now this phenomenon is beyond reach.

Let us close this comparison by returning to the simulation results above. As can be seen, the shapes for $1+a=5$ or 6 are not that round. This is unlike internal diffusion-limited aggregation where even quite small simulations are very close to a ball. A similar phenomenon can be seen in simulations of excited to the center done by Yadin. This is simply indicative that the boundary smoothing mechanism if far less efficient in these models.

3. Pólya's urn

We now move to the main topic of this talk, linearly reinforced random walk, LRRW for short. Naively one might conjecture that, since $\sum \frac{1}{n}$ diverges so slowly, maybe the behaviour of LRRW is not that different from the behaviour of strongly reinforced random walk. Let us start with a simple example that shows that this is not at all the case. The example will be a graph with 3 vertices connected in a line, and x_0 being the middle vertex. The reinforcement function will be $2+n$. With this weight, the process is equivalent to the classic Pólya urn. Recall that in the Pólya urn model one repeatedly takes out a ball, and then returns two balls of the same colour as the ball you took out. Looking at the weights of the LRRW at time $2t$ (i.e. the values of $f(N(e,2t))$), and dividing them by 2, one gets a Pólya urn. Let us recall Pólya's theorem [23]



Theorem 3. *At time t (i.e. when the urn contains $t+1$ balls), the probability that it contains k black balls is exactly $\frac{1}{t}$, for any value of k between 1 and t.*

Proof. Assume inductively that the claim has been proved for $t-1$. Now, to get k black balls at time t you need one of the following:

(1) Have $k-1$ black balls at time $t-1$ and take a black ball out of the bin; or

(2) Have k black balls at time $t-1$ and take a white ball out of the bin.

The probability of (1) is

$$\frac{1}{t-1}\cdot\frac{k-1}{t}$$

where the term $1/(t-1)$ is our inductive assumption, and the term $(k-1)/t$ is the probability to pick a black ball when you have $k-1$ black balls out of a total of t balls. This formula also holds for $k=1$, though for a different reason: here the probability is zero since 0 black balls are not allowed, and the formula indeed gives 0. A similar calculation shows that the probability of (2) is

$$\frac{1}{t-1}\cdot\frac{t-k}{t}.$$

Summing (1) and (2) gives

$$\frac{1}{t-1}\cdot\left(\frac{k-1}{t}+\frac{t-k}{t}\right)=\frac{1}{t}$$

and the induction is complete. □

Thus we see that the picture is very different from that of strongly reinforced random walk. Looking at strongly reinforced random walk at some very large time would give that with probability very close to 1, the walk visited one of the edges a very small number of times, and the other edge all the rest (from symmetry each one gets probability close to $\frac{1}{2}$). Here this is not the case, typically each edge is visited a number of times proportional to t. On the other hand, this is also very different from what we expect for, say, once-reinforced random walk. It is not difficult to analyze this explicitly and see that once-reinforced would have the number of visits to each edge at time t being $\frac{1}{2}t$, with the random errors being of order $\sqrt{t}$. So we see that the linear case is quite special.

The analogy to Pólya's urn works in a number of other cases. Examine the graph $\mathbb{Z}$. Examine a single vertex. Since there are no cycles, you know that when you leave your vertex through some edge e, you are bound to return through e, if you return at all. The other vertices can only effect you through preventing return, not in any other way. Thus you can couple LRRW on $\mathbb{Z}$ to a sequence of i.i.d. Pólya urns. The same holds for any tree, except that you get multicoloured Pólya urns. But these are also well understood. Using these ideas, Pemantle [22] was able to analyze LRRW on regular trees. He showed that there is a phase transition in the initial weight a

Theorem 4. *Let G be a regular tree of degree ≥ 3, and let $a+n$ be the reinforcement function. Then there exists an a_c such that for $a<a_c$ LRRW is (strongly) recurrent, while for $a>a_c$ it is transient.*

Here strong recurrence means that the expected time to return to your starting point is finite. This is significantly stronger than usual recurrence: infinite graphs (like $\mathbb{Z}$ or $\mathbb{Z}^2$) are recurrent but not strongly so (for $\mathbb{Z}$ this is the well-known "gambler's ruin" theorem). In a way it means that the process behaves like random walk on a *finite* graph. Again, compare to the case of once-reinforced random walk on a tree which is transient for any value of the parameter [7].

Theorem 4 shows that importance of the parameter a, and from now on we will simply say "LRRW with a" rather than the cumbersome "LRRW with reinforcement function $a+n$". Note that the parameter a has the opposite effect compared to once-reinforced random walk: there small a gave random-walk-like behaviour, while here it is the large a that give this.

Another case where the theory of urns is applicable is *directed* reinforced random walk. One modifies the definition of N to count directed traversals i.e.

$$N^{\mathrm{dir}}(x,y,t)=\#\{1\leq s\leq t : x_{s-1}=x, x_s=y\}$$

and then again define x_s by taking $a+N^{\mathrm{dir}}$ and normalizing it. Here it is not important what graph one looks at, each vertex can be isolated and treated as a

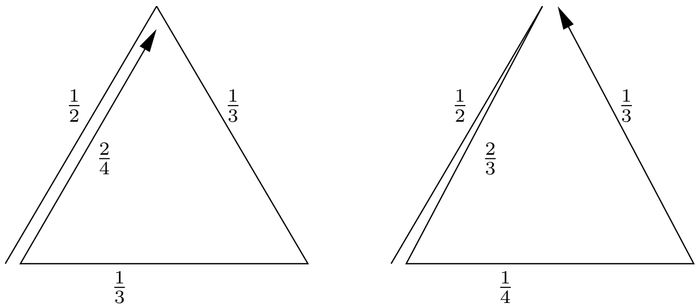


Figure 2. Partial exchangeability

multicoloured urn independent of all others. Nevertheless, this urn argument only reduces the model to a kind of *random walk in random environment*, a notorious problem in its own right. This was tackled on the graphs $\mathbb{Z}^d$ by Enriquez and Sabot [9, 24]. Interestingly, there is no phase transition when $d \geq 3$, the walk is transient for any choice of a.

We will return to Pólya's urn later, but for now let us go back to general graphs.

4. Partial exchangeability

Another way to understand why LRRW is special is that it has a property called *partial exchangeability*. This means that for a given path, its probability depends on how many times each edge is visited, but not on the order in which these visits happen. Figure 2 shows two such paths in a triangle. Both paths have length 4, and both cross one edge twice and the other two edges once, but are different. The numbers in the figure are the probability of each traversal for LRRW with $a = 1$. The reader can verify that the numbers are not the same, but their product is

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{4} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{1}{3}$$

Indeed the numerators on the left are a permutation of the numerators on the right; and ditto for the denominators. The general proof is the same:

Theorem 5. *LRRW is partially exchangeable.*

Proof. We may assume without loss of generality that the reinforcement function is $n + a$ for some $a > 0$. Let γ be a path in the graph G. Then the probability $p(\gamma)$ that $\gamma = (\gamma_0, \ldots, \gamma_l)$ is the beginning of the LRRW path is

$$p(\gamma) = \prod_{i=0}^{l-1} \frac{a + N_\gamma((\gamma_i, \gamma_{i+1}), i)}{\sum_{y \sim \gamma_i} a + N_\gamma((\gamma_i, y), i)}$$

where N_γ is of course the analog of N above for the path γ i.e. $N_\gamma(e, t) = \#\{1 \leq s \leq t : (\gamma_{s-1}, \gamma_s) = e\}$. Now, if $N_\gamma(e, l) = q$ for some q, then the numerators contain the terms a, $a+1,\dots,a+q$ at the q places where e is an edge of γ. This is not specific to linear reinforcement. Now assume that the vertex $v\neq x_0$ appears r times in the path $\gamma_0,\dots,\gamma_{l-1}$ (note that we do not count γ_l). Then the denominators contain $a\deg v+1$, $a\deg v+3,\dots,a\deg v+2r-1$ at the r places where $\gamma_i=v$. For $v=x_0$ you get a similar picture, except the denominators contain $a\deg x_0,\dots,a\deg x_0+2r-2$. In other words it does not matter at which order did γ traverse the edges coming out of v, since each such traversal increases the total weight of the edges surrounding v by 1 (and another 1 is added when returning to the vertex). This is the point which is special to linear reinforcement.

Hence, if you know $N_\gamma(e,l)$ for all e you can reconstruct both numerators and denominators up to a permutation (this requires the observation that you can identify the last vertex of γ, since it is the only one with odd total number of appearances, or x_0 if no vertex has this property). Thus you can calculate $p(\gamma)$ from N_γ, and the process is partially exchangeable. □

Partial exchangeability is interesting because of the following theorem of Diaconis & Freedman [4]

Theorem 6. *Let $x=\{x_0,x_1,\dots\}$ be a random sequence of elements in some state space, which is recurrent. Then x is partially exchangeable if and only if it is a mixture of Markov chains.*

Here recurrent means that the sequence returns to its starting vertex with probability 1. This has to be specified as for general random sequences the various definitions of recurrence are not identical. In fact, the counterexamples in [4] which show that recurrence is necessary consist of two states and a process that gets stuck in the second state. A mixture of Markov chains means that there is some measure μ on the space of Markov chains such that the probability of every event E is given by

$$\int \mathbb{P}^W(E)\,d\mu(W)$$

where W is a Markov chain and $\mathbb{P}^W(E)$ is the probability of E under the Markov chain W. The same concept is known under the name of random walk in random environment. Again, you first pick an environment W randomly, then perform random walk on it. We already mentioned random walk in random environment when we discussed the directed case above. But there each vertex was independent, while in the general case this is typically not the case.

We will not prove the theorem, but just make some remarks on its proof. Recurrence is used to break the infinite process into blocks, from one visit of x_0 to the next. These blocks then turn out to be *fully* exchangeable, i.e. any permutation has the same probability. This allows to apply de Finetti's theorem which states that such a process is a mixture of i.i.d. processes. De Finetti's theorem itself is not difficult either, it is an application of Krein-Milman. The point to note here is that the proof is soft and abstract, without any explicit calculations.

Let us demonstrate the utility of this fact by applying it to Pólya's urn, i.e. to LRRW on a line of length 3 with $a=2$. There is no question as to recurrence, since our walker returns to its starting point every 2 steps without failure. Our random environment can only relate to the central vertex, so asking "what is the random environment?" is equivalent to asking "what is the probability to go left from the central vertex?". In other words, there is only one parameter. Let us denote it by p.

Theorem 7. *The parameter p defined above is uniform in $[0, 1]$.*

Proof. Fix p and examine the Markov chain on a line of length 3 which moves from the central vertex to the left vertex with probability p and to the right vertex with probability $1 - p$. Denote by L_t the number of times the left edge was crossed by time t. By the law of large numbers we know that

$$\lim_{t\to\infty} \frac{L_t}{t} = p \qquad \text{almost surely.}$$

The random walk in random environment tells us that for the measure μ that we seek,

$$\mathbb{P}\left(\frac{L_t}{t} \in E\right) = \int_0^1 \mathbb{P}^p\left(\frac{L_t}{t} \in E\right) d\mu(p).$$

This holds for every measurable E, but let us restrict our attention to intervals. By the explicit solution of Pólya's urn (theorem 3 above), the left hand side is $|E| + O(1/t)$ where $|E|$ is the Lebesgue measure of E. Taking $t \to \infty$ (which can be done inside the integral as μ is a probability measure and the integrand is bounded) gives

$$|E| = \int_0^1 \mathbf{1}\{p \in E\}\, d\mu(p).$$

Since this holds for all intervals E, we get thatμ is uniform, as needed. □

In other words, the theorem of Diaconis and Freedman allows us to get from the "static" description of the Pólya urn we had before, a "dynamic" description which is fuller. We now know that Pólya's urn is equivalent, *as a process*, to choosing some number $p \in [0, 1]$ uniformly and then picking each time a black ball with probability p and a white ball with probability $1 - p$.

In general graphs it proved to be a significant challenge to find an explicit formula for the environment, and this is what we turn to next.

5. The magic formula

Let us consider for a moment finite graphs. Even a triangle would prove to be highly nontrivial! LRRW is recurrent on any finite graph — this is not difficult to show, but we will skip it here. The theorem of Diaconis & Freedman promises us that *some* random environment exists such that LRRW is equivalent to random walk in the environment. But what is it? It turns out there is a formula for it, known fondly as the "magic formula". The history of its discovery is complicated and the reader would be best served by reading about it in detail in [18]. It turns out that the environment is always *reversible* (a fact which does not follow from Diaconis & Freedman). This means that there is a (random) weight function $W : E \to [0, \infty)$ where E is the set of edges of G such that the probability to go from x to y is

$$\frac{W(x,y)}{\sum_{z \sim x} W(x,z)}.$$

It is common to denote the denominator by $W(x)$ and we shall do so below. Reversibility is perhaps the most important property a Markov chain may or may not possess. Most notably a reversible Markov chain gives rise to a *self-adjoint* operator (on l^2 with the weight being W) which is amenably to spectral analysis.

Now, W is not unique since multiplying W by a constant gives rise to exactly the same Markov chain. So we need to normalize and we normalize so that $\sum_{x \sim y} W(x,y) = 1$. For completeness, let us state the magic formula for non-constant initial weights, i.e. assume for every edge e there is some a_e and the transition probabilities at time t are $a_e + N(e,t)$, normalized. The magic formula is then:

Theorem 8. *Let $G = (V, E)$ be a finite graph. Examine LRRW on G, starting from x_0 with initial weights a_e, and let μ be the corresponding random environment. Then the density of μ at a given W is given by*

$$\frac{1}{Z} \cdot \frac{\prod_{e \in E} W(e)^{a_e - 1}}{W(x_0)^{\frac{1}{2}a_{x_0}} \prod_{v \in V \setminus \{x_0\}} W(v)^{\frac{1}{2}(a_v+1)}} \sqrt{\sum_T \prod_{e \in T} W(e)}$$

where the sum inside the square root is a sum over all spanning trees of G; where Z is a normalization constant; and where $a_v = \sum_{e \ni v} a_e$.

The most problematic term is the sum over trees, as it introduces long-range dependencies. By the matrix-tree theorem this sum can be written as a determinant, but that does not simplify the formula significantly.

As an exercise, let us analyse Pólya's urn again:

Second proof of theorem 7. We use the magic formula. All $a_e = 2$, $a_{x_0} = 4$ and for $v \neq x_0$ (i.e. one of the side vertices), $a_v = 2$. Since the normalization is $\sum W(e) = 1$, we have $W(x_0) = 1$. Denote the weight of the left edge by p, so for one of the side vertices $W(v) = p$ while for the other it is $1 - p$. All in all this gives

$$\frac{1}{Z} \cdot \frac{p(1-p)}{1^2 \cdot p^{3/2}(1-p)^{3/2}} \sqrt{p(1-p)}$$

The expression inside the square root is so simple because a line graph has exactly one spanning tree (the graph itself), so the sum is only over one term. Everything cancels and we get that the density is $\frac{1}{Z}$, which of course means that $Z = 1$, proving the theorem. $\square$

Throughout the last decade the magic formula stood in the center of research on LRRW. Even the case of the ladder was not so easy, and was done in [19]. The next result achieved was a tightness result that allowed to show that LRRW is always equivalent to a random walk in random environment, even in the transient case (which is not covered by Diaconis & Freedman) [20]. The most impressive result, also due to Merkl & Rolles [21], was the proof that a certain two-dimensional graph is recurrent. A conjecture that goes back to the 80s [22, page 1241] is that LRRW on $\mathbb{Z}^2$ is recurrent, at least for a sufficiently small. Merkl & Rolles got quite close, but their proofs required to replace each edge by a line of length 130 or more.

Around the same period a surprising connection to certain quantum models appeared, a topic we will discuss next.

6. The hyperbolic sigma model

The hyperbolic sigma model has its roots in physics. As some of the readers (and, frankly, the author) may not be familiar with the necessary background, we will be short and vague. Hopefully the description will still be useful to some. Our starting point is supersymmetry.

Supersymmetry started its life as a branch of particle physics that postulated that each of the known particles has a partner (a "superpartner") which is similar in most aspects, but the superpartner of a fermion is a boson and vice versa. The existence of these particles leads to cancellations which would solve several theoretical problems with the current model. These superpartners were never found in particle accelerators, but the mathematics of these cancellations turned out to be very fruitful.

One potential application of supersymmetry to problems which have nothing to do with bosons or fermions is the Wegner-Efetov approach to random matrices and to certain disordered quantum systems [8]. The titular hyperbolic sigma model captures some of the mathematical difficulties of this approach, and, conjecturally, most of the phenomenology. It has been studied by Disertori, Spencer & Zirnbauer [26, 5, 6]. The model, roughly, is as follows. We are looking at functions σ from the vertices of our graph (say $\mathbb{Z}^d$) into hyperbolic space. The action (or energy) $A(\sigma)$ is a sum over the edges of some interaction *which respects the symmetries of the hyperbolic space.* Then each σ is given probability density proportional to $e^{-\beta A}$ where β is a parameter which is analogous to the initial weights a in LRRW.

The approach taken in [26] to tackle the hyperbolic sigma model is to make a change of variable in the hyperbolic plane (change to the so-called horospherical coordinates) and then integrate some of the variables. The result was a sigma model with values in $\mathbb{R}$ but with a much more complicated interaction term, which included a determinant very similar to the one appearing in the magic formula. This caused significant curiosity but for a few years it was not clear whether the similarity is coincidental or not. This mystery was solved in November 2011.

7. The solution

It was Sabot and Tarrès who managed to make the connection between LRRW and the hyperbolic sigma model rigorous [25]. They investigated a variation on LRRW called the vertex-reinforced jump process, defined in continuous time. On the one hand the vertex-reinforced jump process has a random walk in random environment representation with the density of each environment *identical* to the formulas that appear in the hyperbolic sigma model. This allowed them to harness the results of [5] to show that the vertex-reinforced jump process is recurrent for small a in any graph (the jump process has a parameter which acts analogously to a for LRRW). On the other hand, they showed that LRRW is given by a vertex-reinforced jump process with random weights. Thus, after some modification of the techniques of [5] to work with different, unbounded β in different edges, they were able to show:

Theorem 9. *For every graph G there exists some a_0 such that LRRW is recurrent for all $a < a_0$.*

Simultaneously, with Angel and Crawford, we arrived at the same result [1]. Our approach was completely different and did not use the magic formula at all. The basic idea is very simple.

Basic idea of the proof. Examine some edge $e = (x, y)$ and assume you have traversed e for the first time from x to y. Let f be the edge through which you arrived into x. If a is extremely small, then upon arriving to x you see one edge, f, with weight $1 + a$, and all the others with weight a. Hence you are very likely to return via f. This will repeat on the second, third etc. visit to x. At each time, because f has much larger weight than e (its weight is even increasing, but we will not use that), it will choose to exit x through f multiple times until finally exiting through e. Denote the number of times it exited through f before the first exit through e by N.

We now recall that the process is also a random walk in random environment. The environment might be complicated, but we have earned a bit of information about it. We know that the walker exited the vertex x N times through f before the first time it exited through e. This must mean that the weight of f is approximately N times larger than that of e! In other words, we see that the weight of edges decays exponentially in the "first entrance distance", i.e. if we define f_2 as the edge through which we entered the other vertex of f, f_3 as the edge through which we entered the other vertex of f_2 etc., we finally reach $f_n = x_0$. Then the argument says that the weight of e should decay exponentially in n. $\square$

Comparing to the approach of Disertori-Spencer-Sabot-Tarrs, our approach is much softer, and has almost no explicit calculations. We rely on the existence of a random walk in random environment representation, i.e. on the theorem of Diaconis & Freedman, but that too, as explained above, is a soft result.

Trying to formalize this leads to some minor technical difficulties. The first is that, to get exponential decay, different edge ratios need to be independent.

However, different edges are not independent, because the "first entrance path" is a global property which introduces dependencies between all quantities. To tackle this difficulty we use a somewhat brutish approach and simply count over all such paths, i.e. write

$$\mathbb{E}(W_e) = \sum_\gamma \mathbb{E}(W_e \cdot \mathbf{1}\{\gamma \text{ is the first entrance path}\}).$$

This allows to just fix two edges e and f sharing a vertex x, and define a variable as in the basic idea. These variables (call them $R_{e,f}$) are independent and have expectation $\le C\sqrt{a}$. Thus we argue, very roughly, as follows

$$\begin{aligned}
&\mathbb{E}(W_e \cdot \mathbf{1}\{\gamma \text{ is the first entrance path}\}) = \\
&\qquad \mathbb{E}\left(\prod_{i=1}^{\operatorname{len}\gamma-1} R_{(\gamma_{i-1},\gamma_i),(\gamma_i,\gamma_{i+1})} \cdot \mathbf{1}\{\gamma \text{ is the first entrance path}\}\right) \le \\
&\qquad \mathbb{E}\left(\prod_{i=1}^{\operatorname{len}\gamma-1} R_{(\gamma_{i-1},\gamma_i),(\gamma_i,\gamma_{i+1})}\right) = \prod \mathbb{E}R_{(\gamma_{i-1},\gamma_i),(\gamma_i,\gamma_{i+1})} \le \left(C\sqrt{a}\right)^{\operatorname{len}\gamma}. \qquad (1)
\end{aligned}$$

The first equality is because under the event that γ really is the first entrance path, W_e is really the product of the $R_{e,f}$ along γ. The inequality that follows comes from just throwing this event away. This seems very wasteful, but look at the final result: we show that the expectation is exponentially small, and not only that, we can make the exponent base as small as we want by reducing a. So losing a fixed exponential term by counting over the γ is something that our approach can sustain. The third step in (1) is simply independence, and the fourth is the estimate that came from the "basic idea" sketch above.

Another technical difficulty that might be worth mentioning is that, even though the weights typically decrease along the first entrance path, sometimes they increase. Once you incorporate this phenomenon into the basic sketch, you see that the $R_{e,f}$, despite being typically $C\sqrt{a}$, in fact have infinite expectation. We solve this by taking everything to some small fixed power ($\frac{1}{4}$ worked). Thus (1) as written is false, and what we actually prove is

$$\mathbb{E}\left(W_e^{1/4}\right) \le (C\sqrt{a})^{\operatorname{dist}(e,x_0)}.$$

Returning to our favourite example, Pólya's urn, we see that the phenomenon is real: the ratio between the weights of the two edges has no first moment. So some power must appear in the formulation of the result, though $\frac{1}{4}$ is not optimal. With these two problems solved, we were able to show the exponential decay of weights. Let us state this formally:

Theorem 10. *Let G be a graph with all degrees bounded by K. Then there exists $a_0 = a_0(K) > 0$ such that for all $a \in (0, a_0)$,*

$$\mathbb{E}\left(W_e^{1/4}\right) \le 2K\left(C(K)\sqrt{a}\right)^{\operatorname{dist}(e,v_0)}.$$

8. Where now?

The next step to understand the phase transition is to understand the transient regime. Using our techniques we were able to show the existence of a transient regime on any non-amenable graph. But getting to $\mathbb{Z}^d$ seems to require a new idea.

Next one would like to show that the phase transition is unique, namely that there is some a_c such where the process moves from recurrence to transience, and not, say, that the process switches from recurrence to transience and back a few times before finally becoming transient for high enough a. In many models proving this kind of fact requires monotonicity, but there are also other approaches, so e.g. [10].

There are also many other models we would like to know similar results. We already covered other reinforced walks, but the connection to the hyperbolic sigma models suggests many more questions. Are there other sigma models where one can get an interacting representation for random walk on the generated field? All-in-all this is an exciting and fast-moving field, with a lot more to understand.

References

[1] Omer Angel, Nicholas Crawford and Gady Kozma, *Localization for Linearly Edge Reinforced Random Walks.* Preprint (2012). Available at: arXiv:1203.4010

[2] Amine Asselah and Alexandre Gaudilliere, *Sub-logarithmic fluctuations for internal DLA*. Preprint (2011). Available at: arXiv:1011.4592

[3] Itai Benjamini, Hugo Duminil-Copin, Gady Kozma and Cyrille Lucas. In preparations.

[4] Persi Diaconis and David A. Freedman, *De Finetti's Theorem for Markov Chains.* Ann. Probab. 8:1 (1980), 115–130. Available at: projecteuclid.org

[5] Margherita Disertori and Thomas Spencer, *Anderson Localization for a Supersymmetric Sigma Model.* Commun. Math. Phys. 300:3 (2010), 659–671. Available at: springerlink.com, arXiv:0910.3325

[6] Margherita Disertori, Thomas Spencer and Martin R. Zirnbauer, *Quasi-diffusion in a 3D Supersymmetric Hyperbolic Sigma Model.* Commun. Math. Phys. 300:2 (2010), 435–486. Available at: springerlink.com

[7] Rick Durrett, Harry Kesten and Vlada Limic, *Once edge-reinforced random walk on a tree*. Probab. Theory Related Fields 122:4 (2002), 567–592. Available at: springerlink.com

[8] Konstantin Efetov, *Supersymmetry in Disorder and Chaos.* Cambridge Univ. Press., 1997.

[9] Nathanal Enriquez and Christophe Sabot, *Edge oriented reinforced random walks and RWRE*. C. R. Math. Acad. Sci. Paris 335:11 (2002), 941–946. Available from: sciencedirect.com, jussieu.fr/pageperso/enriquez

[10] Alan Hammond, *Sharp phase transition in the random stirring model on trees.* Preprint (2012). Available at: arXiv:1202.1322

[11] David Jerison, Lionel Levine and Scott Sheffield, *Logarithmic fluctuations for internal DLA*. J. Amer. Math. Soc. 25:1 (2012), 271–301. Available at: ams.org, arXiv:1010.2483

[12] David Jerison, Lionel Levine and Scott Sheffield, *Internal DLA and the Gaussian free field.* Preprint (2011). Available at: arXiv:1101.0596

[13] Gregory Lawler, *Subdiffusive fluctuations for internal diffusion limited aggregation*. Ann. Probab. 23:1 (1995), 71–86. projecteuclid.org

[14] Gregory Lawler, Maury Bramson and David Griffeath, *Internal diffusion limited aggregation*. Ann. Probab. 20:4 (1992), 2117–2140. Available at: projecteuclid.org

[15] Vlada Limic, *Attracting edge property for a class of reinforced random walks*. Ann. Probab. 31:3 (2003), 1615–1654. Available at: projecteuclid.org

[16] Vlada Limic and Pierre Tarrès, *Attracting edge and strongly edge reinforced walks*. Ann. Probab. 35:5 (2007), 1783–1806. Available at: projecteuclid.org

[17] Vlada Limic and Pierre Tarrs, *What is the difference between a square and a triangle?* In and out of equilibrium. 2, 481–495, Progr. Probab., 60, Birkhuser, Basel, 2008. Available at: arXiv:0712.0958

[18] Franz Merkl, Aniko Öry and Silke W. W. Rolles, *The "Magic Formula" for Linearly Edge-reinforced Random Walks*. Statistica Neerlandica 62:3 (2008), 345–363. Available at: wiley.com, tum.de

[19] Franz Merkl and Silke W. W. Rolles, *Edge-reinforced random walk on a ladder*. Ann. Probab. 33:6 (2005), 2051–2093. Available at: projecteuclid.org

[20] Franz Merkl and Silke W. W. Rolles, *Asymptotic Behavior of Edge-reinforced Random Walks*. Ann. Probab. 35:1 (2007), 115–140. Available at: projecteuclid.org

[21] Franz Merkl and Silke W. W. Rolles, *Recurrence of edge-reinforced random walk on a two-dimensional graph*. Ann. Probab. 37:5 (2009), 1679–1714. Available at: projecteuclid.org, arXiv:math/0703027

[22] Robin Pemantle, *Phase Transition in Reinforced Random Walk and RWRE on Trees*. Ann. Probab. 16:3 (1988), 1229–1241. Available at: projecteuclid.org

[23] George Pólya, *Sur quelques points de la théorie des probabilités* [French: On some points of the theory of probability]. Ann. inst. Henri Poincar, 1:2 (1930), 117–161. Available at: numdam.org

[24] Christophe Sabot, *Random walks in random Dirichlet environment are transient in dimension $d \geq 3$*. Probab. Theory Related Fields 151:1-2 (2011), 297–317. springerlink.com, arXiv:0811.4285

[25] Christophe Sabot and Pierre Tarrs, *Edge-reinforced random walk, Vertex-Reinforced Jump Process and the supersymmetric hyperbolic sigma model.* Preprint (2012). Available at: arXiv:1111.3991

[26] Thomas Spencer and Martin R. Zirnbauer, *Spontaneous symmetry breaking of a hyperbolic sigma model in three dimensions*. Commun. Math. Phys. 252:1-3 (2004), 167–187. Available at: springerlink.com, arXiv:math-ph/0410032

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