A nonlinear variational problem in relativistic quantum mechanics

Mathieu LEWIN

Mathieu.Lewin@math.cnrs.fr

(CNRS & Université de Cergy-Pontoise)

joint works with C. Hainzl (Tuebingen), P. Gravejat, É. Séré (Paris) & J.-P. Solovej (Copenhagen)

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Quantum Electrodynamics

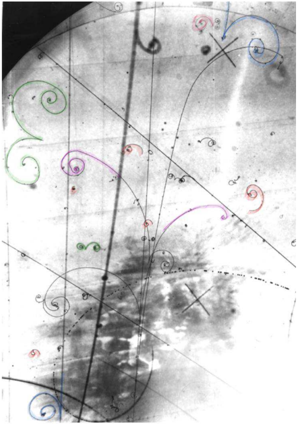
- **Quantum Electrodynamics (QED):** Dirac, Feynman, Schwinger, Tomonaga, Dyson... (1930–50)

matter	$\longrightarrow$ $\longleftarrow$	light
Dirac equation ($\simeq$ relativistic Schrödinger equation)		quantized Maxwell's equations

- Only 2 parameters (m and $\alpha = e^2 = 1/c$) but extremely accurate theory

- **Perturbative theory:**
  - expansion in power series of $\alpha \simeq 1/137$
  - all the terms in the expansion are infinite
  - use of renormalization to remove infinities

The quantum vacuum



Source: `www.cern.ch`

- **The quantum vacuum is not empty!**

Rather a fluctuating medium interacting with its environment

- vacuum polarization in external fields
- electron-positron pair production in strong external fields
- matter = a local perturbation of the vacuum

Dirac operator

- **Energy of a free electron ($p \longleftrightarrow -i\nabla$):**

	non relativistic	relativistic
classical mechanics	$E = p^2/(2m)$	$E^2 = c^2p^2 + m^2c^4$
quantum mechanics	$H = -\Delta/(2m)$	$D^2 = -c^2\Delta + m^2c^4$

- **Dirac operator:**

$$D^0 = -ic\sum_{k=1}^{3}\alpha_k\partial_k + \beta mc^2 = -ic\boldsymbol{\alpha}\cdot\nabla + \beta mc^2$$

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, β = 4×4 hermitian matrices, chosen such that

$$(D^0)^2 = -c^2\Delta + m^2c^4.$$

- Unbounded from below: $\sigma(D^0) = (-\infty; -mc^2] \cup [mc^2; +\infty)$.

Units: $m = c = 1$, $\alpha = e^2$

[ELS08] M.J. Esteban, M.L. and É. Séré. Variational methods in relativistic quantum mechanics. *Bull. Amer. Math. Soc. (N.S.)* **45** (2008), no. 4, 535–593.

Dirac's vacuum

Dirac's interpretation of the negative energies:

> *We make the assumption that, in the world as we know it, nearly all the states of negative energy for the electrons are occupied, with just one electron in each state, and that a uniform filling of all the negative-energy states is completely unobservable to us.* (Dirac, 1934)

- **Dirac's free vacuum:** $P^0_- := \mathbb{1}_{(-\infty,0)}(D^0)$

Our goal: describe **quantum vacuum** in interaction with **classical light**:

- a Dirac equation for a projector of infinite rank describing the vacuum,

coupled to

- the classical Maxwell's equations for the electromagnetic field.

Maxwell's equations in the quantum vacuum

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta\right) A = 4\pi\alpha\big(j_{\rm ext} + j_{\rm vac}\big) \\ \left(\dfrac{\partial^2}{\partial t^2} - \Delta\right) V = 4\pi\alpha\big(\rho_{\rm ext} + \rho_{\rm vac}\big) \\ \nabla \cdot A + \dfrac{\partial}{\partial t} V = 0 \qquad \text{(Lorentz)} \\ \text{Dirac equation for the quantum vacuum} \rightsquigarrow \rho_{\rm vac} \text{ and } j_{\rm vac} \end{cases}$$

- highly nonlinear equations with (conjectured) interesting behavior for large fields (pair production) [HE36,S51]
- divergences $\to$ renormalization

Simplifications:

- stationary case
- purely electrostatic to start with

[HE36] W. Heisenberg & H. Euler (1936). [S51] J. Schwinger, *Physical Review* (1951).

The purely electrostatic case

$$\begin{cases} -\Delta V = 4\pi\alpha\big(\rho_{\rm ext} + \rho_{P-1/2}\big) \\ P = \mathbb{1}_{(-\infty,0)}(D^0 + V) \end{cases}$$

$$\iff \quad P = \mathbb{1}_{(-\infty,0)}\Big(D^0 + \alpha\big(\rho_{\rm ext} + \rho_{P-1/2}\big) * |x|^{-1}\Big) \qquad (*)$$

charge density defined by $\rho_B(x) := \mathrm{tr}_{\mathbb{C}^4}\, B(x,x)$

$(*) \simeq$ an infinite system of coupled nonlinear PDE's of Hartree type

- **No external field case:** $\rho_{\rm ext} \equiv 0$

$$P^0_- - \frac{1}{2} = -\frac{D^0}{2|D^0|} = -\frac{\boldsymbol{\alpha}\cdot p + \beta}{2\sqrt{p^2+1}} \Rightarrow \rho_{P^0_- -1/2} \equiv 0, \quad \text{since } \mathrm{tr}\,\alpha_k = \mathrm{tr}\,\beta = 0$$

So $(P = P^0_-, V = 0)$ is a solution when $\rho_{\rm ext} \equiv 0$

- **With an external field:** $\rho_{\rm ext} \neq 0$

Due to some divergences in Fourier space, no solution to the equation, in any reasonable space [HLS05]

[HLS05] C. Hainzl, M. L. & E. Séré. *J. Phys. A: Math & Gen.* (2005).

Existence with high energy cut off

Π_Λ = orth. projection on $\mathfrak{H}_\Lambda = \{f \in L^2(\mathbb{R}^3;\mathbb{C}^4) \ : \text{supp}(\widehat{f}) \subset B(0;\Lambda)\}$

Theorem (Existence of solutions [HLS05,HLS05'])

Let $0 < \Lambda < \infty$ *and* $\alpha \geq 0$. *For any fixed*

$$\rho_{ext} \in \mathcal{C} = \left\{f \ : \ \widehat{f}(k)|k|^{-1} \in L^2(\mathbb{R}^3)\right\} = \dot{H}^{-1}(\mathbb{R}^3),$$

there exists at least one solution to the equation

$$\begin{cases} P_* = \mathbb{1}_{(-\infty,0)}\big(D_*\big) + \delta \\ D_* = \Pi_\Lambda\big(D^0 + \alpha(\rho_{ext} + \rho_{P_*-1/2}) * |x|^{-1}\big)\Pi_\Lambda \end{cases}$$

with $0 \leq \delta \leq \mathbb{1}_{\{0\}}(D_*)$, *and which is such that*

$$\text{tr}\, P^0_+ P_* P^0_+ + \text{tr}\, P^0_-(1-P_*)P^0_- + \text{tr}(P_* - P^0_-)^2 < \infty \text{ and } \rho_{P_*-1/2} \in \mathcal{C} \cap L^2(\mathbb{R}^3).$$

All these solutions share the same $\rho_{P_*-1/2}$, *hence* $V_* := \alpha(\rho_{ext} + \rho_{P_*-1/2}) * |x|^{-1}$ *is unique.*

If $\alpha \|\rho_{ext}\|_{\mathcal{C}} < \pi^{-1/6} 2^{-11/6}$, *then* $\ker(D_*) = \{0\}$ *hence* $\delta \equiv 0$ *and* P_* *is unique.*

[HLS05'] C. Hainzl, M.L. & E. Séré. *Commun. Math. Phys.* (2005)
[HLS07] C. Hainzl, M.L. & J.P. Solovej. *Commun. Pure Applied Math.* (2007)

A variational proof

- Think of a matrix M. Then $P_* = \mathbb{1}_{(-\infty,0)}(M)$ solves $\inf_{0\leq P\leq 1} \operatorname{tr}(MP)$

- In our case infinite energy, but we can formally subtract that of P^0_-.

So $Q_* = P_* - P^0_-$ (formally) minimizes the relative energy

$$\mathcal{E}(Q) := \operatorname{tr} D^0 Q + \alpha \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_Q(x)\rho_{\rm ext}(y)}{|x-y|}\,dx\,dy + \frac{\alpha}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_Q(x)\rho_Q(y)}{|x-y|}\,dx\,dy$$
$$\geq -\frac{\alpha}{2}\|\rho_{\rm ext}\|^2_{\mathcal{C}}$$

Denoting $Q^{++} = P^0_+ Q P^0_+ \geq 0$ and $Q^{--} = P^0_- Q P^0_- = P^0_-(P-1)P^0_- \leq 0$, etc,

$$\operatorname{tr} D^0 Q = \operatorname{tr} |D^0|(Q^{++} - Q^{--}) \geq 0$$

- Solutions obtained by looking at

$$\inf\left\{\mathcal{E}(Q)\ :\ -P^0_- \leq Q \leq P^0_+,\ Q \text{ Hilbert-Schmidt},\ Q^{\pm\pm} \text{ trace-class},\ \rho_Q \in \mathcal{C}\right\}$$

which is a convex minimization problem.

[CIL] P. Chaix, D. Iracane & P.-L. Lions, *J. Phys. B* (1989)
[BBHS99] V. Bach, J.M. Barbaroux, B. Helffer & H. Siedentop, *Commun. Math. Phys.* (1999)

A formula for $\int \rho$

Recall $Q_* = P_* - P^0_-$.

Theorem (The physical charge [GLS09])

Assume that $\rho_{ext} \in L^1(\mathbb{R}^3) \cap \mathcal{C}$, $\Lambda > 0$, $\alpha \geq 0$.

Then $\rho_{P_*-1/2} = \rho_{Q_*} \in L^1(\mathbb{R}^3)$ *and*

$$\int_{\mathbb{R}^3} \rho_{ext} + \int_{\mathbb{R}^3} \rho_{Q_*} = \frac{\int_{\mathbb{R}^3} \rho_{ext} + \text{``tr''}(Q_*)}{1 + \alpha B_\Lambda}$$

where

$$B_\Lambda = \frac{1}{\pi} \int_0^{\frac{\Lambda}{\sqrt{1+\Lambda^2}}} \frac{s^2 - s^4/3}{1 - s^2} ds = \frac{2}{3\pi} \log \Lambda - \frac{5}{9\pi} + \frac{2\log 2}{3\pi} + O(1/\Lambda^2)$$

and

$$\text{``tr''}(Q_*) = \operatorname{tr} P^0_+ Q_* P^0_+ + \operatorname{tr} P^0_- Q_* P^0_-.$$

For a trace-class operator B, we always have $\operatorname{tr}(B) = \text{``tr''}(B) = \int_{\mathbb{R}^3} \rho_B$.

[GLS09] P. Gravejat, M. L. & E. Séré. *Commun. Math. Phys.* (2009).

Linear response

Take a curve $\mathscr{C}$ enclosing the negative spectrum ($\alpha \ll 1$ for simplicity). Then

$$\begin{aligned} Q_* &= \mathbb{1}_{(-\infty,0)}\Big(D^0 + \alpha(\rho_{\rm ext} + \rho_{Q_*}) * |x|^{-1}\Big) - \mathbb{1}_{(-\infty,0)}\big(D^0\big) \\ &= \frac{1}{2i\pi}\oint_{\mathscr{C}} \left(\frac{1}{z - D^0 - \alpha(\rho_{\rm ext} + \rho_{Q_*}) * |x|^{-1}} - \frac{1}{z - D^0}\right) dz \\ &= \frac{\alpha}{2i\pi}\oint_{\mathscr{C}} \left(\frac{1}{z - D^0}(\rho_{\rm ext} + \rho_{Q_*}) * |x|^{-1}\frac{1}{z - D^0}\right) dz + \textit{higher order terms} \end{aligned}$$

So we have

$$\rho_{Q_*} = \alpha\mathcal{L}(\rho_{Q_*} + \rho_{\rm ext}) + \textit{higher order terms}$$

It turns out that $\widehat{\mathcal{L}(\rho)}(k) = \big(-B_\Lambda + U_\Lambda(k)\big)\widehat{\rho}(k)$ where U_Λ is better behaved

$$\Longrightarrow \rho_{Q_*} + \rho_{\rm ext} = \frac{1}{1 + \alpha B_\Lambda}\rho_{\rm ext} + \cdots$$

Charge renormalization

Choose a nice and small enough $\rho_{\rm ext}$ in $\mathcal{C}$

- Potential without vacuum:

$$V_{\rm ext} = \alpha\rho_{\rm ext} * |x|^{-1} \underset{|x|\to\infty}{\sim} \alpha \frac{\int_{\mathbb{R}^3} \rho_{\rm ext}}{|x|}$$

- Predicted with vacuum:

$$V_* = \alpha(\rho_{\rm ext} + \rho_{P_*-1/2}) * |x|^{-1} \underset{|x|\to\infty}{\sim} \frac{\alpha}{1+\alpha B_\Lambda} \frac{\int_{\mathbb{R}^3} \rho_{\rm ext}}{|x|}$$

The real (observed) α is $\quad \alpha_{\rm ph} := \dfrac{\alpha}{1+\alpha B_\Lambda}$

=screening effect of the quantum vacuum

▶ **Renormalization:** express everything with $\alpha_{\rm ph}$ and study the limit $\Lambda \to \infty$

Problem: we cannot keep $\alpha_{\rm ph}$ fixed, since by definition $\alpha_{\rm ph} B_\Lambda < 1$!

Perturbative renormalization

Idea: expand vacuum polarization in powers of $\alpha_{\rm ph}$, keeping $\alpha_{\rm ph}\log\Lambda$ fixed

Theorem (Perturbative renormalization [GLS11])

For $\epsilon \leq \alpha_{\rm ph} B_\Lambda \leq 1-\epsilon$ and $\alpha_{\rm ph} \ll 1$, we have

$$\left\| V_* - \alpha_{\rm ph}\rho_{ext} * |x|^{-1} - \alpha_{\rm ph}^2 U_{\rho_{ext}} \right\|_{\dot{H}^1(\mathbb{R}^3)} \leq C\alpha_{\rm ph}^3$$

where $U_{\rho_{ext}}$ is the Uehling potential [U35,S35]

$$U_{\rho_{ext}}(x) = \frac{1}{3\pi}\int_1^\infty dt\,(t^2-1)^{1/2}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\int_{\mathbb{R}^3} e^{-2|x-y|t}\,\frac{\rho_{ext}(y)}{|x-y|}\,dy.$$

The result can be generalized to any order [GLS11], leading to a formal series

$$V_* = \sum_{j\geq 1} (\alpha_{\rm ph})^j\, V_j$$

which is probably divergent [D52]

[GLS11] P. Gravejat, M. L. & E. Séré, *Comm. Math. Phys.* (2011)
[S35] R. Serber, *Phys. Rev.* (1935) [U35] E. Uehling, *Phys. Rev.* (1935)
[D52] F. J. Dyson, *Phys. Rev.* (1952)

Back to the electromagnetic case

$$\begin{cases} -\Delta V = 4\pi\alpha\big(\rho_{\rm ext} + \rho_{P-1/2}\big) \\ -\Delta A = 4\pi\alpha\big(j_{\rm ext} + j_{P-1/2}\big) \\ \nabla\cdot A = 0 \\ P = \mathbb{1}_{(-\infty,0)}\big(D^0 - \boldsymbol{\alpha}\cdot A + V\big) \end{cases}$$

where $j_B(x)_k = \mathrm{tr}_{\mathbb{C}^4}\, \alpha_k B(x,x)$

Difficulties:

- gauge invariance
- energy not bounded-below anymore $\rightsquigarrow$ mountain pass of Lagrangian action

$$\mathscr{L}(P,A,V) = \mathrm{tr}\, D^0(P - P^0_-) + \int_{\mathbb{R}^3} V\rho_{P-1/2} - A\cdot j_{P-1/2} + \frac{1}{8\pi\alpha}\int_{\mathbb{R}^3} |\nabla\wedge(A - A_{\rm ext})|^2 - |\nabla(V - V_{\rm ext})|^2$$

The Pauli-Villars regularization

Minimizing w.r.t. P we find a Lagrangian for (A, V)

$$\mathscr{L}(A,V) = \frac{1}{2}\operatorname{tr}\left(|D^0| - |D^0 - \boldsymbol{\alpha}\cdot A + V|\right) + \frac{1}{8\pi\alpha}\int_{\mathbb{R}^3} |\nabla\wedge(A - A_{\rm ext})|^2 - |\nabla(V - V_{\rm ext})|^2$$

Pauli & Villars: 2 fictitious particle fields of high masses $m_1, m_2 \gg 1$ [PV49]

$$\mathscr{L}_{\rm PV}(A,V) = \frac{1}{2}\operatorname{tr}\sum_{j=0}^{2} c_j\left(|D^{0,m_j}| - |D^{0,m_j} - \boldsymbol{\alpha}\cdot A + V|\right) + \frac{1}{8\pi\alpha}\int_{\mathbb{R}^3} |\nabla\wedge(A - A_{\rm ext})|^2 - |\nabla(V - V_{\rm ext})|^2,$$

with $m_0 = c_0 = 1$ and $\sum_{j=0}^{2} c_j = \sum_{j=0}^{2} c_j m_j^2 = 0.$

In [GHLS12]: construction of a mountain pass using tools from convex analysis, for $\rho_{\rm ext}$ and $j_{\rm ext}$ small enough

[PV49] W. Pauli and F. Villars, *Rev. Modern Physics* (1949).
[GHLS12] P. Gravejat, C. Hainzl, M.L. & É. Séré, preprint `arXiv` (2012)

Conclusion and open problems

- **Conclusions:**
  - Constructed (regularized) stationary solutions for electromagnetic field in Dirac's vacuum (small external fields in the magnetic case)
  - Odd mathematical properties related to charge renormalization
  - Renormalization rigorously established order by order
  - Several mathematical tools to handle variational problems with an infinite rank orthogonal projection. Turned out to be useful in other situations, e.g. in [CDL08,HLS12,FLLS12]

- **Perspectives / work in progress:**
  - Time-dependent equations
  - Stationary and dynamical pair production in strong fields (first results in stationary case in [S11])

[CDL08] É. Cancès, A. Deleurence & M.L., *Commun. Math. Phys.* (2008)
[HLS12] C. Hainzl, M.L. & C. Sparber, preprint `arXiv` (2012)
[FLLS12] R.L. Frank, M.L., E.H. Lieb & R. Seiringer, *Duke Math. J.* (2012)
[S11] J. Sabin, preprint `arXiv` (2011)