

Combinatorial Heegaard Floer Theory

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Motivation (4D)

Much insight into smooth 4-manifolds comes from PDE techniques: *gauge theory* and *symplectic geometry*.

- **Donaldson** (1980's): used the Yang-Mills equations to get constraints on the intersection forms of smooth 4-manifolds, etc. Solution counts \longrightarrow *Donaldson invariants*, able to detect exotic smooth structures on closed 4-manifolds.
- **Seiberg-Witten** (1994): discovered the monopole equations, which can replace Yang-Mills for most applications, and are easier to study. Solution counts \longrightarrow *Seiberg-Witten invariants*.
- **Ozsváth-Szabó** (2000's): developed Heegaard Floer theory, based on counts of holomorphic curves in symplectic manifolds. Their *mixed HF invariants* of 4-manifolds are conjecturally the same as the Seiberg-Witten invariants, and can be used for the same applications (in particular, to detect exotic smooth structures).

Motivation (3D)

All three theories (Yang-Mills, Seiberg-Witten, Heegaard-Floer) also produce invariants of closed 3-manifolds, in the form of graded Abelian groups called *Floer homologies*. These have applications of their own, e.g.:

- What is the minimal genus of a surface representing a given homology class in a 3-manifold? (**Kronheimer-Mrowka, Ozsváth-Szabó**)
- Does a given 3-manifold fiber over the circle? (**Ghiggini, Ni**)

In dimension 3, the Heegaard-Floer and Seiberg-Witten Floer homologies are known to be isomorphic: work of **Kutluhan-Lee-Taubes** and **Colin-Ghiggini-Honda**, based on the relation to **Hutchings's** ECH.

There also exist Floer homologies for knots (or links) $K \subset S^3$. Applications: knot genus, fiberedness of the knot complement, concordance, unknotting number, etc.

Motivating problem

All these gauge-theoretic or symplectic-geometric invariants are defined by counting solutions of nonlinear elliptic PDE's. As such, they are very difficult to compute.

Question

Are the (Yang-Mills, Seiberg-Witten, Heegaard-Floer) invariants of knots, 3-manifolds, and 4-manifolds *algorithmically computable*? In other words, do they admit *combinatorial* definitions?

The answer is now **Yes** for the Heegaard-Floer invariants (mod 2).

This is based on joint work with / work of (various subsets of) : **Robert Lipshitz, Peter Ozsváth, Jacob Rasmussen, Sucharit Sarkar, Andrés Stipsicz, Zoltan Szabó, Dylan Thurston, Jiajun Wang.**

Summary of Heegaard Floer theory

Ozsváth-Szabó (2000)

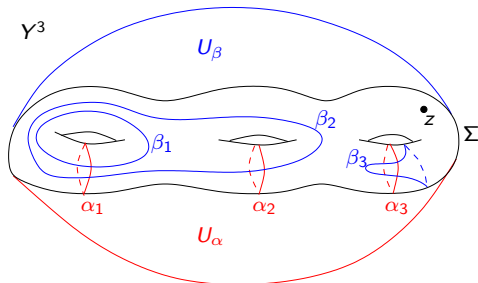
Y^3 closed, oriented 3-manifold $\rightarrow HF^-(Y), \widehat{HF}(Y), HF^+(Y) =$ modules over the polynomial ring $\mathbb{Z}[U]$ (variants of Heegaard Floer homology).

Start with a marked Heegaard diagram for Y :

- $\Sigma =$ surface of genus g
- $\alpha = \{\alpha_1, \dots, \alpha_g\}$ collection of disjoint, homologically linearly independent, simple closed curves on Σ , specifying a handlebody U_α
- $\beta = \{\beta_1, \dots, \beta_g\}$ a similar collection, specifying U_β
- a basepoint $z \in \Sigma - \cup \alpha_i - \cup \beta_i$

such that $Y = U_\alpha \cup_\Sigma U_\beta$.

A marked Heegaard diagram



$$U_\alpha = \Sigma \cup \bigcup_{i=1}^g D_i^\alpha \cup B^3, \quad \partial D_i^\alpha = \alpha_i$$

$$U_\beta = \Sigma \cup \bigcup_{i=1}^g D_i^\beta \cup B^3, \quad \partial D_i^\beta = \beta_i.$$

Heegaard Floer homology

Consider the tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$$

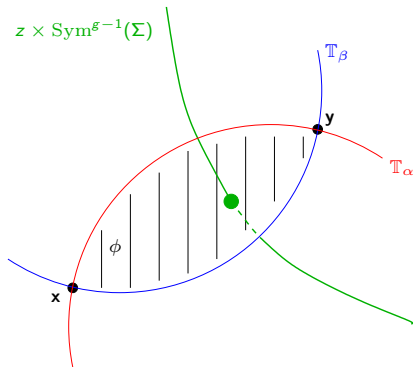
$$\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$$

inside the symmetric product $\text{Sym}^g(\Sigma) = (\Sigma \times \cdots \times \Sigma)/S_g$, viewed as a symplectic manifold.

The Heegaard Floer complex $CF^-(Y)$ is freely generated (over $\mathbb{Z}[U]$) by intersection points

$$\mathbf{x} = \{x_1, \dots, x_g\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$$

with $x_i \in \alpha_i \cap \beta_j \subset \Sigma$. The differential is given by counting pseudo-holomorphic disks in $\text{Sym}^g(\Sigma)$ with boundaries on $\mathbb{T}_\alpha, \mathbb{T}_\beta$. These are solutions to the nonlinear Cauchy-Riemann equations, and depend on the choice of a generic family J of *almost complex structures* on the symmetric product.



$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y})} n(\phi, J) \cdot U^{n_z(\phi)} \mathbf{y},$$

where $\pi_2(\mathbf{x}, \mathbf{y})$ is the space of relative homology classes of Whitney disks from \mathbf{x} to \mathbf{y} , and $n(\phi, J) \in \mathbb{Z}$ is the signed count of holomorphic disks in the class ϕ .

The variable U keeps track of the quantity $n_z(\phi)$, the intersection number between ϕ and the divisor $\{z\} \times \text{Sym}^{g-1}(\Sigma)$.

One can show that $\partial^2 = 0$. We define:

- $HF^- = H_*(CF^-)$;
- $\widehat{HF} = H_*(CF^- / (U = 0))$;
- $HF^+ = H_*(U^{-1}CF^- / CF^-)$.

Ozsváth and Szabó proved that these are invariants of the 3-manifold Y . Typically, \widehat{HF} has less information than HF^- and HF^+ (although it suffices for most 3D applications).

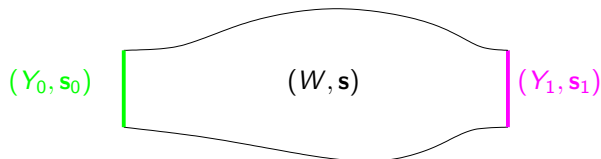
For example, $HF^-(S^3) = \mathbb{Z}[U]$, $\widehat{HF}(S^3) = \mathbb{Z}$, $HF^+(S^3) = \mathbb{Z}[U^{-1}]$.

All three variants ($\circ = +, -, \text{hat}$) split according to Spin^c -structures on Y :

$$HF^\circ(Y) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} HF^\circ(Y, \mathfrak{s})$$

Heegaard Floer theory as a TQFT

The three variants of Heegaard Floer homology are functorial under Spin^c -decorated cobordisms.



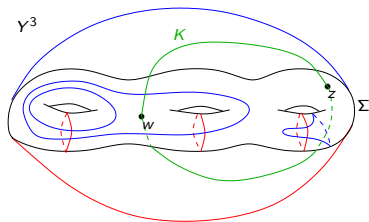
$$F_{W,s}^\circ : HF^\circ(Y_0, s_0) \longrightarrow HF^\circ(Y_1, s_1)$$

Combining F^+ and F^- , Ozsváth and Szabó constructed their *mixed HF invariants* $\Psi_{X,s}$ of closed 4-manifolds with $b_2^+(X) > 1$. These invariants can detect exotic smooth structures.

Going back to a marked Heegaard diagram $(\Sigma, \alpha, \beta, z)$, if one specifies another basepoint w , this gives rise to a knot $K \subset Y$.

Ozsváth-Szabó, Rasmussen (2003)

Counting pseudo-holomorphic disks and keeping track of the quantities $n_z(\phi)$ and $n_w(\phi)$ in various ways yields different versions of *knot Floer homology* $HFK^\circ(Y, K)$.



One can also break the knot (or link) into more segments, by using more basepoints.

Simplest version of knot Floer homology: $\widehat{HFK}(S^3, K)$

We count only pseudo-holomorphic disks with $n_z(\phi) = n_w(\phi) = 0$:

$$\widehat{HFK}(S^3, K) = \bigoplus_{m, s \in \mathbb{Z}} \widehat{HFK}_m(S^3, K, s)$$

Its Euler characteristic is the Alexander polynomial of the knot:

$$\sum_{m, s \in \mathbb{Z}} (-1)^m q^s \cdot \text{rk } \widehat{HFK}_m(S^3, K, s) = \Delta_K(q).$$

The genus of a knot

$$g(K) = \min\{g \mid \exists \text{ embedded, oriented, } \Sigma^2 \subset S^3 \text{ of genus } g, \partial\Sigma = K\}$$

can be read from \widehat{HFK} (**Ozsváth-Szabó**, 2004):

$$g(K) = \max\{s \geq 0 \mid \exists m, \widehat{HFK}_m(S^3, K, s) \neq 0\}.$$

More about $\widehat{HFK}(S^3, K)$

In particular:

K is the unknot ($g(K) = 0$) $\iff \widehat{HFK}(S^3, K) \cong \mathbb{Z}_{(0,0)}$.

By a result of **Ghiggini** (2006), knot Floer homology has enough information to also detect the right-handed trefoil, the left-handed trefoil, and the figure-eight knot.

Ni (2006) showed that $S^3 \setminus K$ fibers over the circle if and only if $\bigoplus_m \widehat{HFK}_m(S^3, K, g(K)) \cong \mathbb{Z}$.

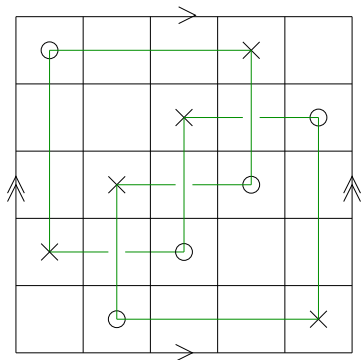
Theorem (M.-Ozsváth-Sarkar, 2006)

All variants of Heegaard Floer homology for links $L \subset S^3$ are algorithmically computable.

Every link in S^3 admits a *grid diagram*; that is, an n -by- n grid in the plane with O and X markings inside such that:

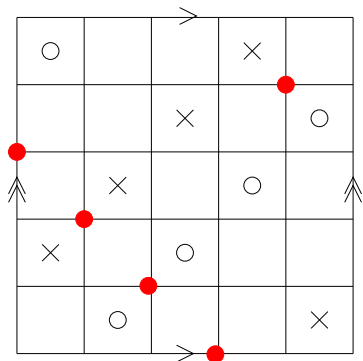
- Each row and each column contains exactly one X and one O ;
- As we trace the vertical and horizontal segments between O 's and X 's (verticals on top), we see the link L .

A grid diagram for the trefoil



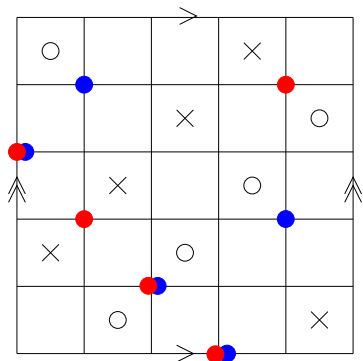
Grid diagrams are particular examples of Heegaard diagrams: if we identify the opposite sides to get a torus, we let: α = horizontal circles, β = vertical circles; the X 's and O 's are the basepoints.

Generators

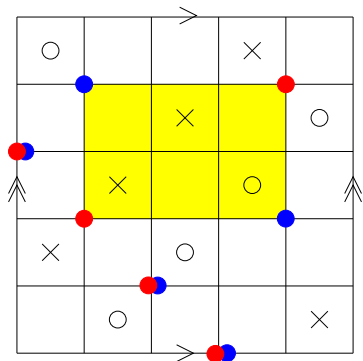


The generators $\mathbf{x} = \{x_1, \dots, x_n\}$ are n -tuples of points on the grid (one on each vertical and horizontal circle).

Generators



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In a grid diagram, isolated pseudo-holomorphic disks are in 1-to-1 correspondence to empty rectangles on the grid torus (that is, having no red or blue dots inside).

Algorithm for computing *HF*K

Start with a grid diagram G . Form a complex $C(G)$ freely generated by n -tuples (permutations), with differential

$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{r \in \text{Rect}^\circ(\mathbf{x}, \mathbf{y})} \epsilon(r) \cdot \mathbf{y},$$

where Rect° is the set of all *totally empty* rectangles from \mathbf{x} to \mathbf{y} (no O or X markings inside). The sign $\epsilon(r) \in \{0, 1\}$ was fixed in (**M.-Ozsváth-Szabó-D.Thurston**, 2006).

$$H_*(C(G)) \cong \widehat{HF}K(S^3, K) \otimes (\mathbb{Z}_{(-1,-1)} \oplus \mathbb{Z}_{(0,0)})^{\otimes(n-1)}$$

We get a nice combinatorial algorithm for detecting the genus of a knot and, in particular, if the knot is unknotted.

Why does this work?

Key fact: torus admits a metric of nonnegative curvature. (It could also work on a sphere.)

How about HF for 3-manifolds?

A typical 3-manifold does not admit a Heegaard diagram of genus 0 or 1. (Only S^3 , $S^1 \times S^2$ and lens spaces do.)

However, on a surface of higher genus we can move all negative curvature to a neighborhood of a point \rightarrow algorithm for computing \widehat{HF} of any 3-manifold (**Sarkar-Wang**, 2006), based on *nice diagrams*.

Similarly, one can compute the cobordism maps $\widehat{F}_{W,s}$ for any simply connected W (**Lipshitz-M.-Wang**, 2006). These suffice to detect exotic smooth structures on some 4-manifolds with boundary, but not on any closed 4-manifolds.

Other related results:

- One can give combinatorial proofs of invariance for knot Floer homology, even over \mathbb{Z} (**M.-Ozsváth-Szabó-D.Thurston**, 2006) and for \widehat{HF} of 3-manifolds (**Ozsváth-Stipsicz-Szabó**, 2009).
- Alternate combinatorial descriptions of knot Floer homology: **Ozsváth-Szabó** (2007), **Baldwin-Levine** (2011); and of $\widehat{HF}(Y^3)$: **Lipshitz-Ozsváth-D.Thurston** (2010).

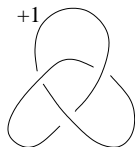
To get algorithms for all HF invariants, we need to present 3- and 4-manifolds in terms of links in S^3 ...

Surgery presentations of 3-manifolds

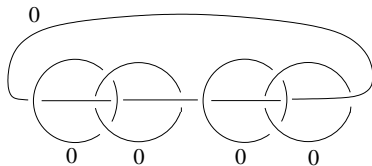
Any closed 3-manifold is integral surgery on a link in S^3
(**Lickorish-Wallace**, 1960):

$$Y = (S^3 \setminus \text{nbhd}(L)) \cup_{\phi} (\text{nbhd}(L)).$$

For example,



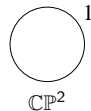
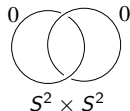
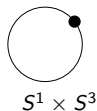
is the Poincaré sphere



is $S^1 \times \Sigma_2$

Kirby diagrams of 4-manifolds

By Morse theory, a closed 4-manifold can be broken into a 0-handle, some 1-handles (dotted circles), some 2-handles (numbered circles), some 3-handles (automatic), and a 4-handle. For example:



Theorem (Link Surgery Formula: M.-Ozsváth, 2010)

If $L \subset S^3$ is a link, HF^+ , \widehat{HF} , and (a completed version of) HF^- of integral surgeries on L can be expressed in terms of Floer complexes associated to L and its sublinks, together with maps relating them. The maps on Floer homology induced by 2-handle attachments (surgeries) can also be understood from link Floer data.

Theorem (M.-Ozsváth-D.Thurston, 2009)

All (completed) versions of HF for 3-manifolds are algorithmically computable (mod 2). So are the mixed invariants $\Psi_{X,s}$ (mod 2) for closed 4-manifolds X with $b_2^+(X) > 1$ and $\mathbf{s} \in \text{Spin}^c(X)$.

Thus, in principle one can now detect exotic smooth structures on 4-manifolds combinatorially.

Algorithm: represent the 3-manifold (or 4-manifold) in terms of a link, take a grid diagram for the link, and use the Link Surgery Formula.

We know that isolated holomorphic disks correspond to empty rectangles. To understand the maps between different link Floer complexes, one also needs to understand higher polygon counts combinatorially.

For closed 4-manifolds, not very effective (the $K3$ surface needs a grid of size at least 88).

For 3-manifolds and some 4-dimensional cobordisms (e.g. surgeries on knots in S^3), hard but doable.

For knots and links in S^3 , very useful!

Computer programs (**Baldwin-Gillam**, **Beliakova-Droz**) can calculate HF of knots with grid number up to 13.

Open problems

- 1 Develop more efficient algorithms.
- 2 Extend the algorithms to the integer-valued invariants (rather than mod 2).
- 3 Give combinatorial proofs of invariance.
- 4 Prove the Seiberg-Witten / Heegaard-Floer equivalence in 4D.
- 5 Understand the relationship of Heegaard-Floer theory to Yang-Mills (instanton) theory, and to π_1 .
- 6 Use the (links \rightarrow 3-manifolds \rightarrow 4-manifolds) strategy to turn other link invariants into 3- and 4-manifold invariants.