# On blow-up curves for semilinear wave equations

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# The equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1} u, \\ \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $p > 1$, $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $u_0 \in \mathrm{H}^1(\mathbb{R})$ and $u_1 \in \mathrm{L}^2(\mathbb{R})$.
The Cauchy problem in $H^1 \times L^2(\mathbb{R})$ is locally wellposed (Ginibre and Velo, Lindblad and Sogge) and we have finite speed of propagation.

**Maximal solution in $\mathrm{H}^1(\mathbb{R}) \times \mathrm{L}^2(\mathbb{R})$**

- either it exists for all $t \in [0, \infty)$ (**global solution**),
- or it exists for all $t \in [0, \bar{T})$ (**singular solution**).

**Existence of singular solutions (Levine)**

*if* $\int_{\mathbb{R}} \left( \frac{1}{2}(u_1)^2 + \frac{1}{2}(\partial_x u_0)^2 - \frac{1}{p+1}|u_0|^{p+1} \right) dx < 0$, *then $u$ is not global*.

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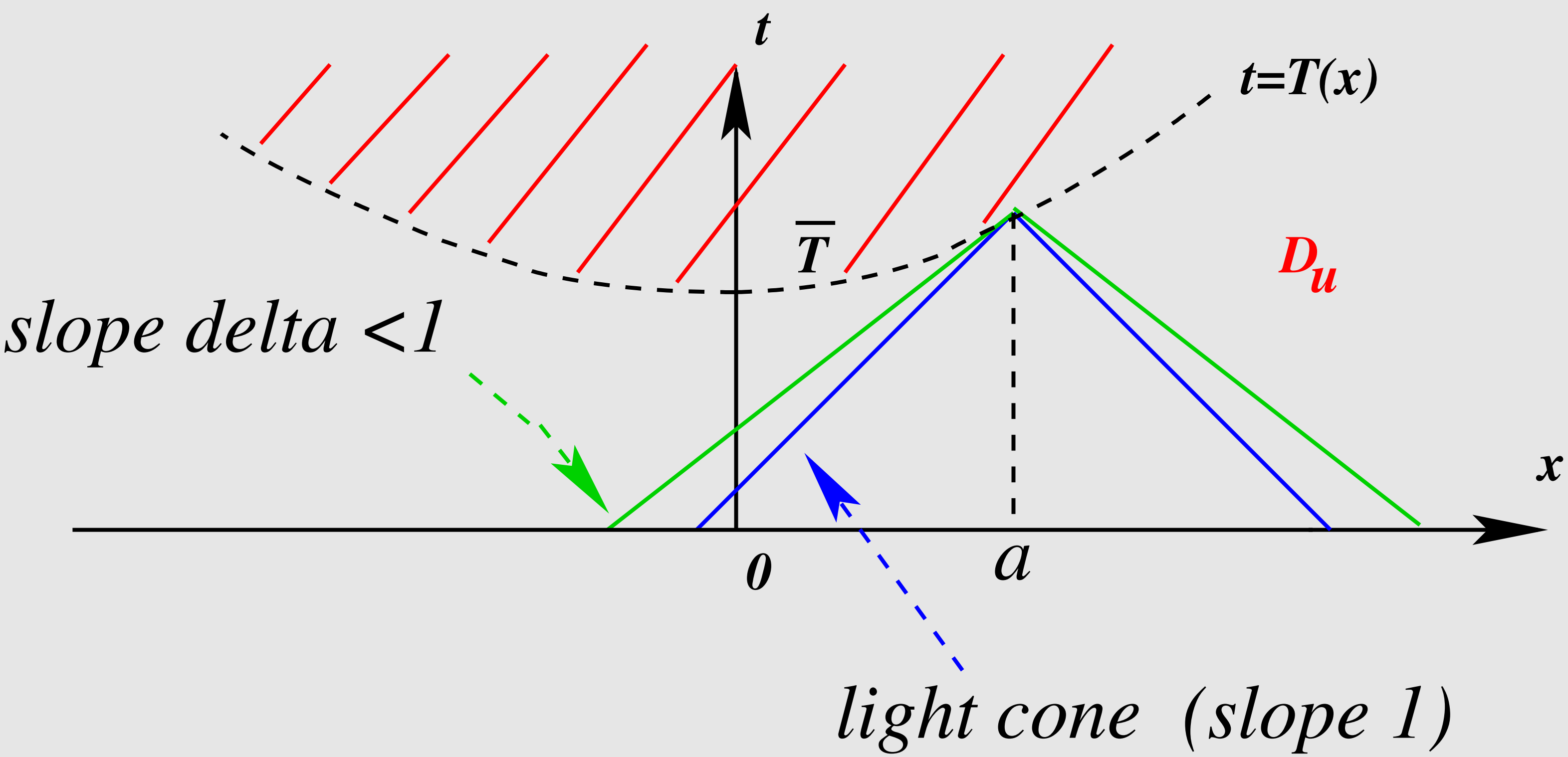
# Singular solutions: the maximal influence domain, characteristic points




For all $x \in \mathbb{R}$, there exists a “local” blow-up time $T(x)$ ($\bar{T} = \inf T(x)$ is the **blow-up time**).

A point $a$ is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$. The point is said *characteristic* if not.
$\mathcal{R} \subset \mathbb{R}$ is the set of *non* characteristic points ($\mathcal{S} \subset \mathbb{R}$ of characteristic points).

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# Known results, for an arbitrary solution

- The blow-up set $\Gamma = \{(x, T(x))\}$ is 1-Lipschitz.

- $\mathcal{R} \neq \emptyset$ (Indeed, $\bar{x}$ such that $T(\bar{x}) = \min_{x \in \mathbb{R}} T(x)$ is non characteristic).

- Caffarelli and Friedman (1985 and 1986) had a criteria to have $\mathcal{R} = \mathbb{R}$, $x \mapsto T(x)$ of class $C^1$ and ODE blow-up (using the **positivity of the fundamental solution**):
under conditions on initial data that ensure that for some $\delta_0 > 0$,

$$u \geq 0 \text{ and } \partial_t u \geq (1 + \delta_0)|\partial_x u|.$$

**The aim of this talk**:
- To describe precisely the blow-up set, and the solution near the blow-up set, *for an arbitrary blow-up solution*. Is the example of Caffarelli and Friedman gives the full picture.

# Questions and new results

▷ **Existence**
- Are there characteristic points? *yes,* $\mathcal{S} \neq \emptyset$.

▷ **Regularity**
- Is $\mathcal{R}$ open? *yes*
- Is $\Gamma$ of class $C^1$ ? *yes on* $\mathcal{R}$
- "How is" $\mathcal{S}$? *isolated points*
- How does $\Gamma$ look like near $\mathcal{S}$? *corner shaped*

▷ **Asymptotic behavior (profile)**
- How does the solution behave near a non characteristic point? *we have the profile*
- and near a characteristic point? *we have a precise decomposition into solitons*

▷ **Construction**
- *All* the possibilities described just above *do* occur.

## Part 1: Existence of characteristic points

**Prop. (M.Z.)** There exist *initial data which give solutions with a characteristic point.*

**Example**: We take odd initial data.

- Then, if the solution blows up then the origin is a characteristic point.

- If we perturb the constructed initial data, then the new solution blows up and has a characteristic point near zero.

- In addition, if (locally) the solution is positive, there are no characteristic points.

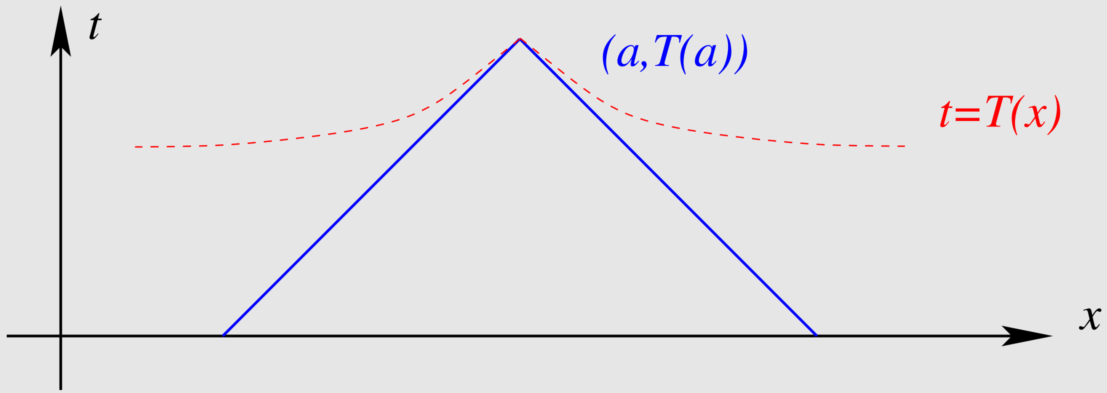
# Regularity of the blow-up set

**Th. (M.Z.)** *The set of non characteristic points* $\mathcal{R}$ **is open** *and* $T(x)$ **is of class** $C^1$ *on this set.*

**Th. (M.Z.)** *The set of characteristic points* $\mathcal{S}$ *is made of* **isolated points**. *Moreover,*

$$T(x) - T(a) + |x-a| \sim \frac{C_0^{\pm}|x-a|}{|\log|x-a||^{\gamma(a)}} \text{ as } x \to a^{\pm}, \qquad (1)$$

where $\gamma(a) = \frac{(k(a)-1)(p-1)}{2}$ with $k(a) \in \mathbb{N}, k(a) \geq 2$.

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# Blow-up speed or the $L^\infty$ norm behavior

**Cor.**
(i) **(Case of non-characteristic points)** *If $x_0 \in \mathcal{R}$, then*

$$\frac{(T(x_0)-t)^{-\frac{2}{p-1}}}{C} \le \sup_{|x-x_0|<T(x_0)-t} |u(x,t)| \le C(T(x_0)-t)^{-\frac{2}{p-1}}$$

(i) **(Case of characteristic points)** *If $x_0 \in \mathcal{S}$, then*

$$\frac{|\log(T(x_0)-t)|^{\frac{k(x_0)-1}{2}}}{C(T(x_0)-t)^{\frac{2}{p-1}}} \le \sup_{|x-x_0|<T(x_0)-t} |u(x,t)| \le \frac{C|\log(T(x_0)-t)|^{\frac{k(x_0)-1}{2}}}{(T(x_0)-t)^{\frac{2}{p-1}}}.$$

*where $k(x_0) \ge 2$ is the solitons' number in the decomposition of $w_{x_0}$.*

**Rk.**
When $x_0 \in \mathcal{R}$, the speed is given by the associated ODE $u'' = u^p$.
When $x_0 \in \mathcal{S}$, the speed is higher. It has a $\log$ correction depending on the number of solitons.

# Idea of the proof

The techniques are based on

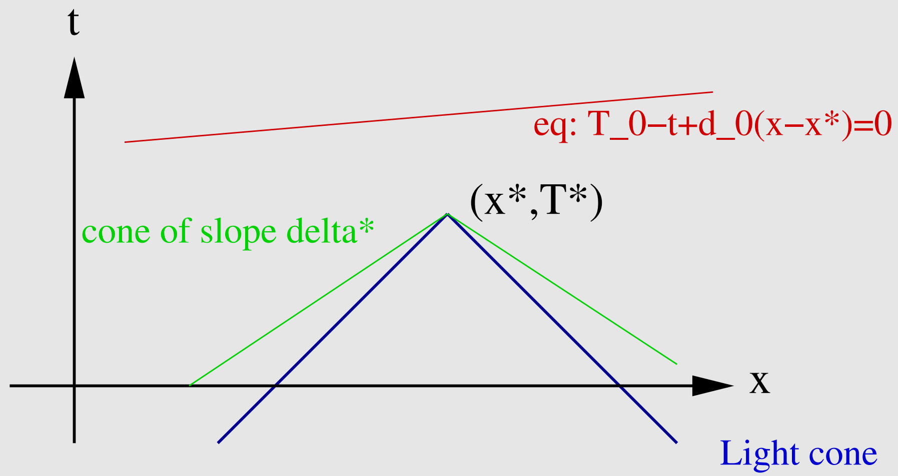
- ▷ - a very good understanding of the **behavior of the solution in selfsimilar variables in the energy space** related to the selfsimilar variable (see Part 3 of this talk).
- ▷ - a **Liouville Theorem** (see next slide).

# A Liouville Theorem

**Th. (M.Z.)** Consider $u(x,t)$ a solution of $u_{tt} = u_{xx} + |u|^{p-1}u$ such that:

- $u$ is defined in the *infinite* green cone and $u$ is less than $(T^* - t)^{-\frac{2}{p-1}}$ (to avoid periodic in space solution). Then,
- either $u \equiv 0$,
- or there exists $T_0$, $d_0$ and $\theta_0 = \pm 1$ such that $u$ is defined below the red line by

$$u(x,t) = \theta_0 \kappa_0(p) \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x - x^*))^{\frac{2}{p-1}}}.$$



# Part 3: Selfsimilar variable

## Selfsimilar transformation for all $x_0 \in \mathbb{R}$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

$(x,t)$ in the light cone of vertex $(x_0, T(x_0)) \Longleftrightarrow (y,s) \in (-1,1) \times [-\log T(x_0), \infty)$.

**Equation on $w = w_{x_0}$**: For all $(y,s) \in (-1,1) \times [-\log T(x_0), \infty)$:

$$\partial^2_{ss} w - \tfrac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \tfrac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\tfrac{p+3}{p-1}\partial_s w - 2y\partial^2_{sy} w$$

where $\rho(y) = (1-|y|^2)^{\frac{2}{p-1}}$

# Blow-up profile at a non characteristic point

**Th. (M.Z.)** *There exist* $C_0 > 0$ *and* $\mu_0 > 0$ *such that*
*if* $x_0$ *is* **non characteristic**, *then there exist* $d(x_0) \in (-1, 1)$, $e(x_0) = \pm 1$ *such that:*
*(i) For all* $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

*and* $E(w_{x_0}(s) \to E(\kappa_0)$ *where the energy space (ii)* $d(x_0) = T'(x_0)$.

# Asymptotic behavior at a *characteristic* point

**Th. (M.Z. )** If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $k(x_0) \geq 2$, $e(x_0) = \pm 1$ and $d_i(s) = -\tanh \zeta_i(s)$ such that:
(i)

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

(ii) Introducing $\zeta_i(s) = -\tanh^{-1} d_i(s)$, we get

$$\frac{1}{c_1} \zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} \text{ and } \zeta_i(s) = \left(i - \frac{(k+1)}{2}\right) \frac{(p-1)}{2} \log s + C_i$$

(ii) $E(w_{x_0}(s)) \to k(x_0) E(\kappa_0)$ as $s \to \infty$.

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# Asymptotic behavior at a *characteristic* point (cont.)

**Rk.**
- As $s \to \infty$, $w_{x_0}$ becomes like a **decoupled** sum of *equidistant* stationary solutions (“solitons”), with *alternate* signs.
- The main difficulty in the proof is to prove that $k(x_0) \geq 2$ (the case $k(x_0) = 0$ is harder to eliminate).
- As a conscequence, we have an energy criterion for non characteristic points: If

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then $x_0 \in \mathcal{R}$.

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# Construction of blow-up modalities in the characteristic case

**Th. (Côte-Zaag)**: Given $k \geq 2$ and $\zeta_0 \in \mathbb{R}$, there exists a solution $u(x,t)$ such that $0 \in \mathcal{S}$ and

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_0(s) \end{pmatrix} - \sum_{i=1}^{k} (-1)^i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

with $d_i(s) = -\tanh \zeta_i(s)$, $\zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0$ and $(\bar{\zeta}_i(s))_i$ is THE solution of

$$\frac{1}{c_1} \bar{\zeta}_i'(s) = e^{-\frac{2}{p-1}(\bar{\zeta}_i - \bar{\zeta}_{i-1})} - e^{-\frac{2}{p-1}(\bar{\zeta}_{i+1} - \bar{\zeta}_i)} \text{ with } \bar{\zeta}_1(s) + \cdots + \bar{\zeta}_k(s) \equiv 0.$$

**Remark**: There is version with $\{x_n\}$ a discrete subset of $\mathbb{R}$.

# A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_{-1}^{1} \left( \frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2(1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy,$$

[a]o a Hardy-Sobolev inequality, well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \;\mid\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left( q_1^2 + (\partial_y q_1)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

**Lemma 1 (Monotonicity (Antonini-Merle))** *For all $s_1$ and $s_2$:*

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1} \int_{s_1}^{s_2} \int_{-1}^{1} (\partial_s w)^2 (1-|y|^2)^{\frac{2}{p-1}-1} dyds.$$

# An upper bound in selfsimilar variables

**Prop. (M.Z.)** For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-1}^{1} \left( \frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2(1-|y|^2) + \frac{(p+1)}{(p-1)^2} w^2 + \frac{1}{p+1}|w|^{p+1} \right) \rho dy \leq K.$$

Idea of the proof of the upper bound are
- Selfsimilar transformation and existence of a Lyapunov functional
- Gagliardo-Nirenberg estimates in the energy space $\mathcal{H}$.

All stationary solutions of the $w$ equation in $\mathcal{H}$ are
either $w \equiv 0$ or there exist $d \in (-1,1)$ and $e = \pm 1$ such that $w(y) = e\kappa(d,y)$
where

$$\forall (d,y) \in (-1,1)^2, \ \ \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}.$$

**Remark**: We have 3 connected components. $E(0) = 0 < E(\pm\kappa(d)) = E(\kappa_0)$.

# Asymptotic behavior at a *non characteristic* point

Take $x_0 \in \mathbb{R}$ **non characteristic**. Using a covering argument for $x$ near $x_0$, we obtain that $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1\times L^2(-1,1)}$ is bounded. Similar techniques will hold in $\mathcal{H}$ for characteristic point using the critically of this space with respect to the dispersion.

**Question**: Does $w_{x_0}(y,s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$). In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible**. See for similar difficulty and approach, results for the $L^2$ **critical KdV** (Martel and Merle, Martel,M., Raphael), $L^2$ **critical NLS** (Merle and Raphaël).

- The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need **a modulation technique**.
- The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

# Idea of the proof of the results in the *characteristic* case

The results are: the decomposition into solitons, the corner property and the fact that $\mathcal{S}$ is made of isolated points.

4 main steps are needed:

- Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons, with no information on the signs or the distance between the solitons' centers.
- Step 2: Characterization of the case $k(x_0) \geq 2$. Proof of *the upper bound* in the corner property if $k(x_0) \geq 2$.
- Step 3: Excluding the case $k(x_0) = 0, 1$ if $x_0 \in \partial\mathcal{S}$ and $\partial\mathcal{S} = \mathcal{S}$.
- Step 4: We prove that $\mathcal{S}$ is made of isolated points.

## Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons

By similar dispersive argument than in non characteristic case, critically of $\mathcal{H}$ for the dispersive relation, we have

**Prop.** *There exist $k(x_0) \geq 0$ and continuous $d_i(s) \in (-1,1)$ such that*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \sum_{i=1}^{k(x_0)} e_i(x_0) \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

*with $\zeta_{i+1}(s) - \zeta_i(s) \to \infty$ as $s \to \infty$ and $d_i(s) = -\tanh \zeta_i(s)$.*

**Rk.**

- $k(x_0) = 0, 1$ is not excluded. The sum is finite from the fact that the energy of all stationary solution is $E(\kappa_0)$ together with the fact that the energy of the solution is bounded , then the above sum is $0$.
- We have no information on the signs $e_i(x_0)$.
- We have no equivalent for $\zeta_i(s)$ as $s \to \infty$.

# Step 2: Case $k(x_0) \geq 2$; A differential equation on the solitons' centers

Here, we assume that $k(x_0) \geq 2$ (we don't prove that fact here).
Linearizing the equation in the $w(y,s)$ setting around the sum of the solitons, and using again the dispersive effect to justify the next order of appriximation, we get the following system on the solitons' centers in the $\xi$ variable:
for all $i = 1, ..., k$ and $s$ large enough, we have

$$\frac{1}{c_1}\zeta_i' \sim -e_{i-1}e_i e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} + e_i e_{i+1} e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}.$$

Since for all $i$, we have $\zeta_{i+1}(s) - \zeta_i(s) \to \infty$ as $s \to \infty$, using ODE techniques, we find that

$$e_i e_{i+1} = -1 \text{ and } \zeta_i(s) \sim \left(i - \frac{k(x_0)+1}{2}\right)\frac{(p-1)}{2}\log s.$$

The upper bound on the blow-up rate gives the *upper bound* in the corner property.

## Step 3: Excluding the case where $x_0 \in \mathcal{S}$ and $k(x_0) = 0, 1$

A good understading of the *non-characteristic* case is *crucial*. By contradiction,

- if $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 0$ with a minimal property, then $\|w_{x_0}(s)\|_{\mathcal{H}} \to 0$ as $s \to \infty$. and for a $x_1 \in \mathcal{R}$ near $x_0$ and $s_1$, $E(w_{x_1}(s_0)) \leq \frac{1}{2}E(\kappa_0)$. Contradiction.

- The case $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 1$ for again a minimal element of this type is again delicate. we first use a traping result on the parameter $d$, its geometrical interpretation to conclude a contradiction.

- We conclude that the interior of $\mathcal{S}$ is empty using the previous results and the decomposition of the solution. Contradiction.

In particular, we show that for all $a \in \mathcal{S}$, we have $k(a) \geq 2$.

# Step 4: $\mathcal{S}$ is made of isolated point

Consider $a \in \mathcal{S}$. We would like to prove that for $x \sim a$ and $x \neq a$, $x \in \mathcal{R}$.

Goal: Prove

$$\|w_x(s^*) \pm \kappa(d^*)\|_{\mathcal{H}} \leq \epsilon_0,$$

for $s^* = |\log|x-a|| + L$ and $d^* = d^*(x)$ and $L$ large. Thus,

$$E(w_x(s^*)) \leq \frac{3}{2}E(\kappa_0)$$

and $x$ is non-characteristic.

For that, we introduice $\kappa_1^*(d, \pm e^s)$ the heteroclinic orbits connecting $\kappa(d)$ to $0$ or to $\infty$, where

$$\kappa_1^*(d,\nu,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy+\nu)^{\frac{2}{p-1}}}.$$

3 Steps are needed. The main difficulty is that $w_x(y,s)$ depends on $T(x)$, which is an unknown in the problem.

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# Step 4: $\mathcal{S}$ is made of isolated point (cont.)

- (*Initialization*):
Decomposition of $w_a$ for $s$ large and continuity arguments to show that for small $\epsilon > 0$ and $\bar{L} > 0$

$$\|w_x(\bar{L}) - e\sum_{i=1}^{k}(-1)^{i+1}\kappa_1^*(d_i(\bar{L}),\mu_i(\bar{L}))\|_{\mathcal{H}} \leq \epsilon.$$

- (*Stability of the decomposition of $w_x$*):
Stability of this decomposition is stable in time (until the centers are not "far") implies that we propagate this decomposition from $s = \bar{L}$ to $s = |\log|x-a|| + L$

$$\sup_{\bar{L}\leq s\leq|\log|x-a||+L}\left\|w_x(s) - e\sum_{i=1}^{k}(-1)^{i+1}\kappa_1^*\left(d_i(s),\mu_i(s)\right)\right\|_{\mathcal{H}} \to 0$$

as $\bar{L}\to\infty$, $L\to\infty$ and $x\to a$, for some parameters $(d_i(s),\mu_i(s))$.

# Step 4: $\mathcal{S}$ is made of isolated point (cont.)

- (*Using information coming from the decomposition of* $w_a(y,s)$):
From the two decomposition of $w_x$ and $w_a$ in the intersection of the backward light cones with vertexes $(a, T(a))$ and $(x, T(x))$and the fact that they have to agree up to error terms

$$\left\| \sum_{i=2}^{k} (-1)^{i+1} \kappa_1^* \left( d_i(s), \mu_i(s) \right) \right\|_{\mathcal{H}} \to 0$$

for

$$s = |\log|x-a|| + L.$$

The conclusion follows.