Random maps and continuum random 2-dimensional geometries

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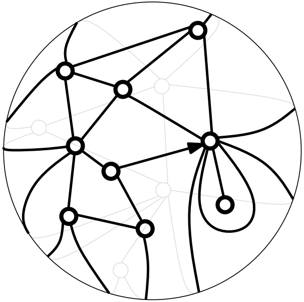
Plane maps

Definition

A plane map is an embedding of a connected, finite (multi)graph into the 2-dimensional sphere, considered up to orientation-preserving homeomorphisms of the sphere.



$V(\mathbf{m})$ Vertices
$E(\mathbf{m})$ Edges
$F(\mathbf{m})$ Faces
$d_{\mathbf{m}}(u, v)$ graph distance

A rooted map: distinguish one oriented edge.

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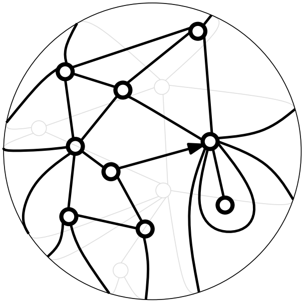

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Motivation

- Maps are seen as discretized 2D Riemannian manifolds.
- This comes from 2D quantum gravity, in which a basic object is the partition function

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} [\mathcal{D}g] \exp(-\alpha \, \mathrm{Area}_g(M))$$

  - M is a 2-dimensional orientable manifold,
  - $\mathcal{R}(M)$ is the space of Riemannian metrics on M,
  - $\mathrm{Diff}^+(M)$ the set of orientation-preserving diffeomorphisms,
  - $\mathcal{D}g$ is a "Lebesgue" measure on $\mathcal{R}(M)$ invariant under the action of $\mathrm{Diff}^+(M)$. This, and the induced measure $[\mathcal{D}g]$, are the problematic objects.

How to deal with $[\mathcal{D}g]$?

One can replace

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} [\mathcal{D}g] \longrightarrow \sum_{T \in \mathbf{Tr}(M)} \delta_T$$

where $\mathbf{Tr}(M)$ is the set of triangulations of M.

- Then one tries to take a scaling limit of the right-hand side, in which triangulations approximate a “smooth”, continuum surface.
- Analog to path integrals, in which random walks can be used to approximate Brownian motion.
- The success of this approach comes from the rich literature on enumerative theory of maps, after Tutte’s work or the literature on matrix integrals.
- However, metric aspects of maps could only be dealt with recently, using bijective approaches.
- Another approach: Liouville quantum gravity (Polyakov, David, Duplantier-Sheffield...).

How to choose a random map

All maps we consider are rooted.

- pick a p-angulation with n vertices, uniformly at random (ex $p = 3$ triangulation, $p = 4$ quadrangulation)
- From now on we only consider bipartite plane maps (with faces of even degree), mostly for technical simplicity.
- Boltzmann distribution: let $w = (w_k, k \geq 1)$ be a non-negative non-zero sequence, $w_1 < 1$. Define a measure by

$$\mathbb{B}_w(\mathbf{m}) = \prod_{f \in F(\mathbf{m})} w_{\deg(f)/2} , \qquad \mathbf{m} \text{ rooted, bipartite}$$

- Let

$$\mathbb{B}_w^n(\cdot) = \mathbb{B}_w\Big(\cdot \, \Big| \, \{\mathbf{m} \text{ with } n \text{ vertices}\}\Big) ,$$

defining a probability measure. Uniform on $2p$-angulations with n vertices if $w_k = \delta_{kp}$.

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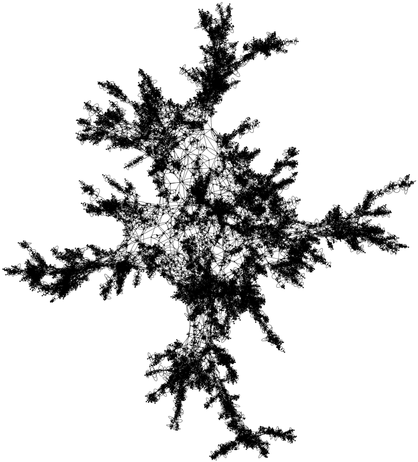
Simulation of a uniform random plane quadrangulation with 30000 vertices, by J.-F. Marckert



- Q_n uniform random variable in the set $\mathbf{Q}_n$, of rooted plane quadrangulations with n faces (law $\mathbb{B}_{\delta_2}^{n+2}$)
- The set $V(Q_n)$ of its vertices is endowed with the graph distance d_{Q_n}.
- Typically $d_{Q_n}(u, v)$ scales like $n^{1/4}$ (Chassaing-Schaeffer (2004)).

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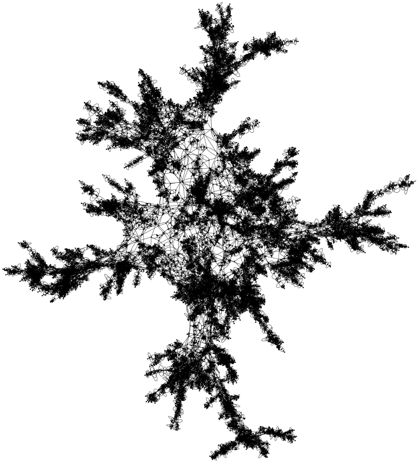

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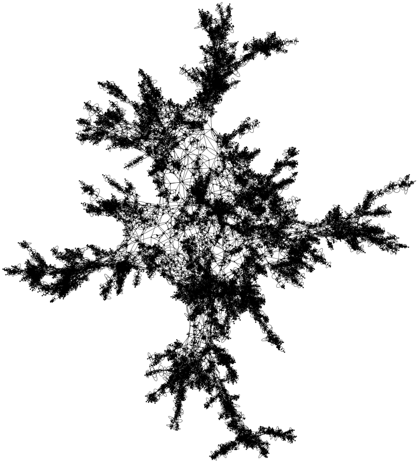

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Convergence to the Brownian map

Theorem

There exists a random metric space (S, D^*)*, called the Brownian map, such that the following convergence in distribution holds*

$$(V(Q_n), (8n/9)^{-1/4} d_{Q_n}) \xrightarrow[n\to\infty]{(d)} (S, D^*)$$

as $n \to \infty$*, for the Gromov-Hausdorff topology.*

- This result has been proved independently by Le Gall (2011) and Miermont (2011), *via* different approaches. Also universality results in Le Gall (2011).
- Before this work, convergence was only known up to extraction of subsequences, but the uniqueness of the limiting law was open.

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Some of the previous results on random maps

- Chassaing-Schaeffer (2004)
  - identify $n^{1/4}$ as the proper scaling and
  - compute limiting functionals for random quadrangulations.
- Generalized by Marckert-M. (2007), M. (2008) to the larger class of Boltzmann random maps.
- Marckert-Mokkadem (2006) introduce the Brownian map.
- Le Gall (2007)
  - Gromov-Hausdorff tightness for rescaled $2p$-angulations
  - the limiting topology is the same as that of the Brownian map.
  - all subsequential limits have Hausdorff dimension 4
- Le Gall-Paulin (2008), and later M. (2008) show that the limiting topology is that of the 2-sphere.
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The Cori-Vauquelin-Schaeffer bijection: coding maps with trees

- Let $\mathbf{T}_n$ be the set of rooted plane trees with n edges,
- $\mathbb{T}_n$ be the set of labeled trees $(\mathbf{t}, \mathbf{l})$ where $\mathbf{l} : V(\mathbf{t}) \to \mathbb{Z}$ satisfies $\mathbf{l}(\mathrm{root}) = 0$ and

$$|\mathbf{l}(u) - \mathbf{l}(v)| \leq 1\,, \qquad u, v \text{ neighbors}\,.$$

Theorem (Cori-Vauquelin 1981, Schaeffer)

The construction to follow yields a bijection between $\mathbb{T}_n \times \{0, 1\}$ and $\mathbf{Q}_n^*$, the set of rooted, pointed plane quadrangulations with n faces.

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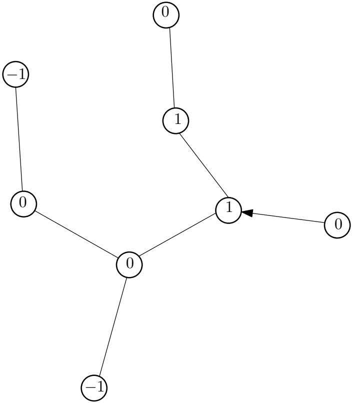

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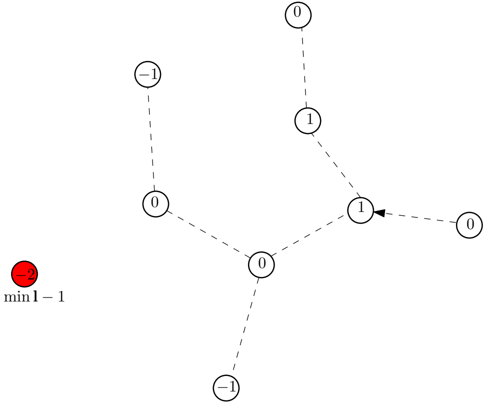

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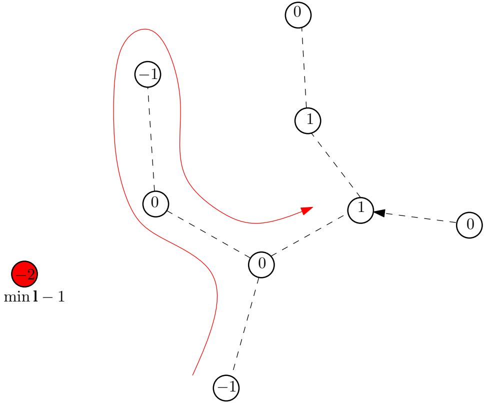

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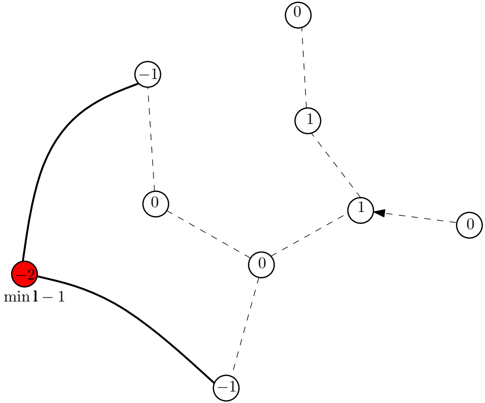

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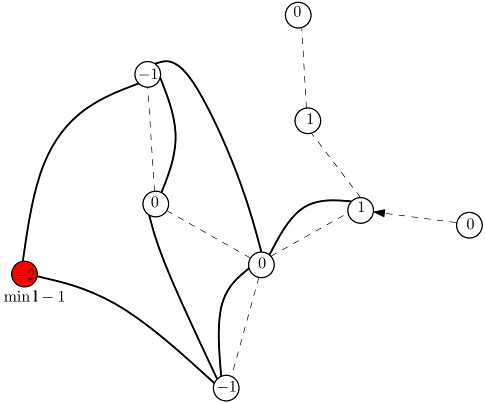

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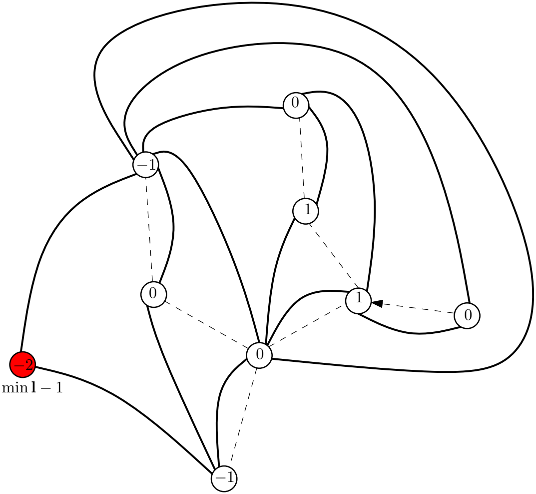

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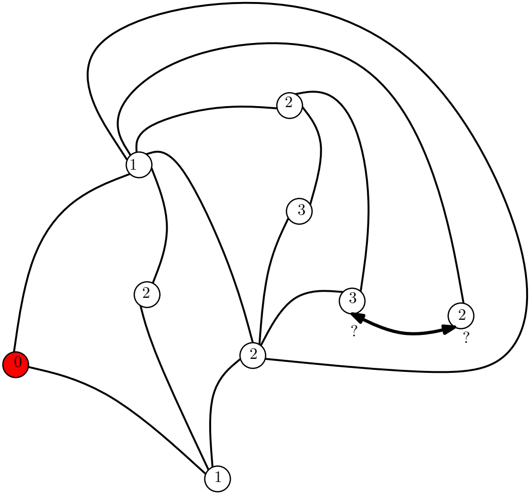


Note that the labels are geodesic distances in the map. Key formula:

$$d_{\mathbf{q}}(v_*, v) = \mathbf{l}(v) - \inf \mathbf{l} + 1$$

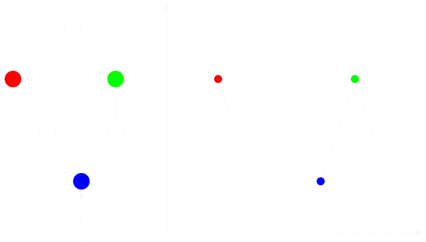
Scaling limits for plane trees: Aldous' CRT

- The Brownian tree arises as the scaling limit of many discrete random tree models, e.g. uniform random element T_n of $\mathbf{T}_n$:

$$(V(T_n), (2n)^{-1/2} d_{T_n}) \to \mathcal{T} \, ,$$

for the Gromov-Hausdorff distance.

- Note that a tree with n edges can be encoded by a walk (Harris encoding): let $u_i, 0 \leq i \leq 2n$ be the $i+1$-th explored vertex in contour order (started at the root). Let C_i the height of u_i.



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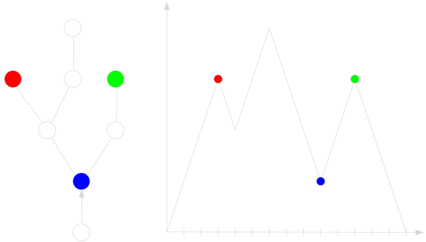

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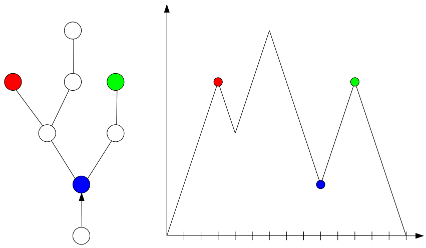

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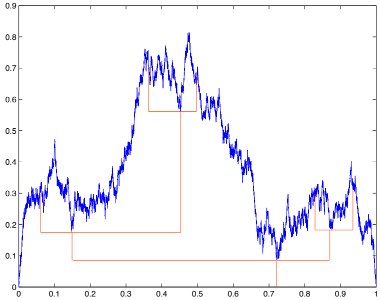
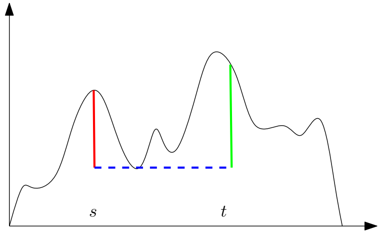
The Brownian CRT

- Let T_n be uniform in $\mathbf{T}_n$, and C^n be its contour process. As $n \to \infty$, the process $((2n)^{-1/2} C^n_{[2nt]}, 0 \leq t \leq 1)$ converges in distribution to a normalized Brownian excursion $(\mathbb{e}_t, 0 \leq t \leq 1)$.
- Define

$$d_{\mathbb{e}}(s,t) = \mathbb{e}_s + \mathbb{e}_t - 2 \inf_{s \wedge t \leq u \leq s \vee t} \mathbb{e}_u .$$

This is a pseudo-distance on $[0,1]$. The continuum random tree is the quotient space $\mathcal{T}_{\mathbb{e}} = [0,1]/\sim_{\mathbb{e}}$, where $s \sim t \iff d_{\mathbb{e}}(s,t) = 0$. It defines an $\mathbb{R}$-tree.





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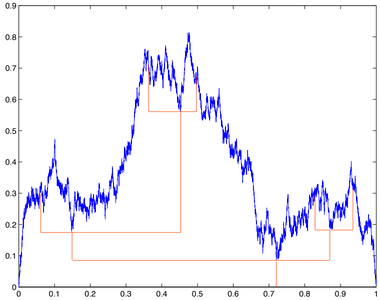

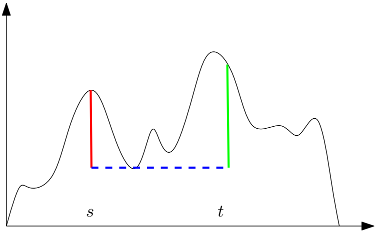

Brownian labels on the Brownian tree

- Once the tree is build, one can consider a white noise supported by the tree, or, equivalently, branching Brownian paths.
- Informally, we let Z be a centered Gaussian process run on $\mathcal{T}$, with covariance function

$$\mathrm{Cov}\left(Z_a, Z_b\right) = d_{\mathcal{T}}(\mathrm{root}, a \wedge b)\,,$$

$a \wedge b$ the most recent common ancestor of a, b.

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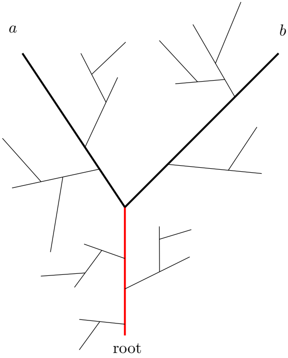

Convergence of labeled trees

- Let (T_n, ℓ_n) be uniform in $\mathbb{T}_n$. Then

$$\left(\frac{1}{\sqrt{2n}} T_n, \left(\frac{9}{8n}\right)^{1/4} \ell_n\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_{\mathbb{e}}, Z)\,,$$

e.g. in the sense of convergence of contour encoding functions.

- We want to apply to $(\mathcal{T}_{\mathbb{e}}, Z)$ a similar construction as the CVS bijection.
- The Brownian map is a quotient of $\mathcal{T}_{\mathbb{e}}$ by the equivalence relation generated by

$$\{(a,b) : Z_a = Z_b = \min_{[a,b]} Z\}\,,$$

where $[a,b]$ is the interval from a to b around $\mathcal{T}_{\mathbb{e}}$.

- The resulting quotient set is endowed with a distance D such that

$$D(a,a^*) = Z_a - \inf Z$$

if $a \in \mathcal{T}_{\mathbb{e}}$ and $a^* = \mathrm{argmin}(Z)$. Other distances $D(a,b)$, $a, b \neq a_*$?

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$$D(a, a^*) = Z_a - \inf Z$$

  if $a \in \mathcal{T}_{\mathbb{e}}$ and $a^* = \mathrm{argmin}(Z)$. Other distances $D(a, b)$, $a, b \neq a_*$?

Convergence of labeled trees

- Let (T_n, ℓ_n) be uniform in $\mathbb{T}_n$. Then
$$\left(\frac{1}{\sqrt{2n}} T_n, \left(\frac{9}{8n}\right)^{1/4} \ell_n\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_{\mathbb{e}}, Z),$$
e.g. in the sense of convergence of contour encoding functions.
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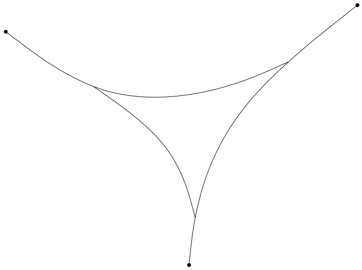
Shape of the typical geodesics

- An important point is to describe precisely the geodesic γ between two "generic" points x_1, x_2. More precisely, one must show that it is a patchwork of small segments of geodesic paths headed toward a_* (geodesics tend to stick).
- So we want to show that Γ, the set of points x on γ from which we can start a geodesic to a_* not meeting γ again, is a small set.



Proposition

There exists $\delta \in (0,1)$ such that a.s. for every $\varepsilon > 0$, the set Γ can be covered with less than $\varepsilon^{-(1-\delta)}$ D-balls of radius ε. In particular $\dim_{\mathcal{H}}(\Gamma) < 1$.

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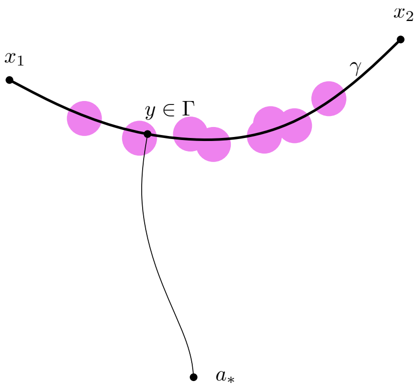

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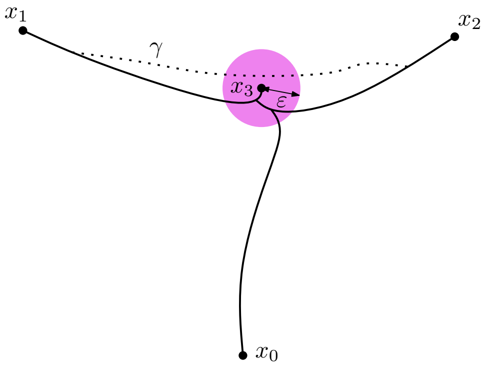
Idea of proof: Quickly separating geodesics



- A method to prove the main proposition is to approach points of Γ by points where geodesics perform a quick separation: Evaluate the probability that for 4 randomly chosen points x_0, x_1, x_2, x_3,
  - The three geodesics from x_3 to x_0, x_1, x_2 are disjoint outside of the ball of radius ε around x_3
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Proposition (codimension estimate)

The probability of the latter event is bounded above by $C\varepsilon^{3+\chi}$ for some $\chi > 0$.

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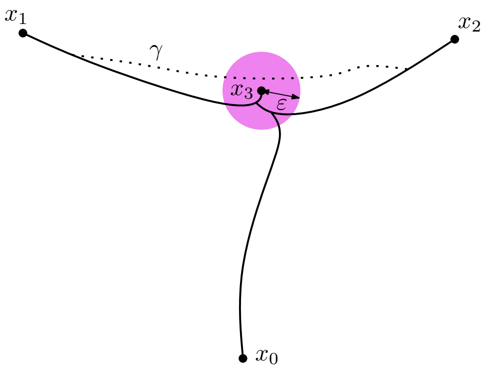

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Brownian map and stable maps universality classes

For Boltzmann maps $\mathbb{B}_w^n$ sampled accoding to "generic" sequences of weights, the Brownian map still prevails in the limit.

Theorem (Le Gall 2011)

If $(w_k, k \geq 1)$ is a weight sequence with finite support, then if M_n has law $\mathbb{B}_w^n$, there exists a constant b_w such that $(V(M_n), b_w n^{-1/4} d_{M_n})$ converges in distribution to the Brownian map.

But under certain conditions called non-generic, implying in particular that $w_k \sim C\rho^k k^{-a}$ for some $a \in (3/2, 5/2)$, the limit is different.

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For non-generic weights, if M_n has law $\mathbb{B}_w^n$, the sequence $(V(M_n), n^{-1/(2a-1)} d_{M_n})$ converges in distribution, at least along some extraction, to a random metric space (S_a, d_a) with Hausdorff dimension a.s. equal to $2a - 1$, the stable map with index a.

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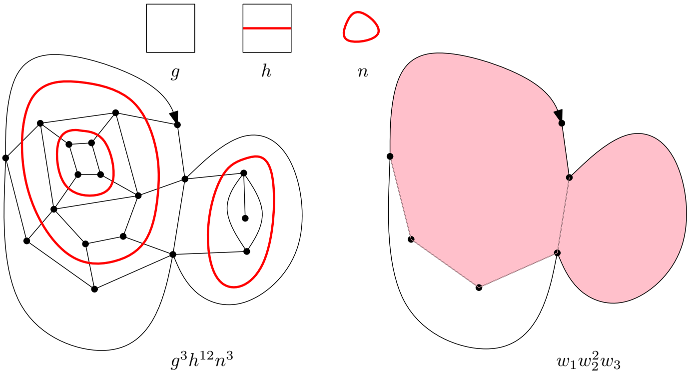
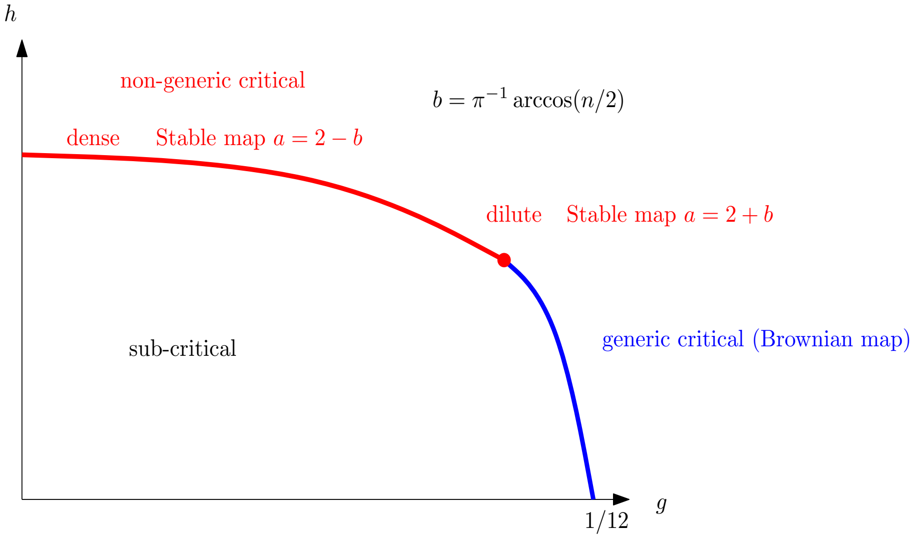
Non-genericity in gaskets of loop models



- Decorate a quadrangulation with simple and mutually avoiding dual loops: weight $W^{(n)}_{g,h}(\mathbf{q}) = g^{\#quad} h^{|loops|} n^{\#loops}$
- Emptying the interior of the loops, gives a Boltzmann random map

$$w_k = g\delta_{k2} + nh^{2k} \sum_{|\partial \mathbf{q}|=2k} W^{(n)}_{g,h}(\mathbf{q}) .$$

Phase diagram at fixed $n \in (0, 2]$ [Borot-Bouttier-Guitter 2011]



Some future directions and open questions

- Topology of stable maps (in progress):
  - if $a \in [2, 5/2)$ then (S_a, d_a) is a random Sierpinsky carpet (holes have simple, mutually avoiding boundaries).
  - if $a \in [2, 5/2)$ then holes have self and mutual intersections.
- This phenomenon recalls the phases of Schramm-Loewner Evolutions and Conformal Loop Ensembles (Sheffield-Werner). This is no coincidence, as CLEs are the conjectured limits of $O(n)$ loop models on regular lattices.
- Is there a “canonical embedding” of the Brownian map in $\mathbb{S}^2$?
- Is there a canonical embedding of stable maps into CLE's?