Commuting higher rank ordinary differential operators

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5 July 2012

Commuting ordinary differential operators

- Introduction: commuting ordinary differential operators of rank one
- Commuting higher rank ordinary differential operators
- Evolution equations of Krichever–Novikov type
- Open problems

Introduction

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} u_i(x)\partial_x^i, \quad L_m = \partial_x^m + \sum_{i=0}^{m-2} v_i(x)\partial_x^i.$$

There is a classification of commuting operators (I.M. Krichever)

Examples

1. u_i, v_j are constant
2. $L_n = F_1(L),\ L_m = F_2(L)$, F_1, F_2 are polynomials

Wallenberg, 1903:

1. $n = 1$, $L_2 = F(L_1)$.
2. $n = 2, m = 3$.

$$L_1 = \partial_x^2 + u(x), \qquad L_2 = \partial_x^3 + \frac{3}{2}u(x)\partial_x + \frac{3}{4}u'(x),$$

where $u(x)$ satisfies the equation

$$(u')^2 + 2u^3 + s_1 u + s_0 = 0.$$

Introduction

Theorem (Schur, 1905)

If $L_1 L_2 = L_2 L_1$ and $L_1 L_3 = L_3 L_1$ ($L_1 \neq const$), then

$$L_2 L_3 = L_3 L_2.$$

Theorem (Burchnall, Chaundy, 1923)

If $L_1 L_2 = L_2 L_1$, then there exist a non-trivial polynomial $R(\lambda, \mu)$ of two commuting variables such that $R(L_1, L_2) = 0$.

Examples

- For Wallenberg's operators

$$L_1 = \partial_x^2 + u(x), \quad L_2 = \partial_x^3 + \frac{3}{2}u(x)\partial_x + \frac{3}{4}u'(x),$$

$$(u')^2 + 2u^3 + s_1 u + s_0 = 0,$$

the polynomial R has the form

$$R(z,w) = w^2 - \left(z^3 + \frac{s_1}{8}z - \frac{s_0}{8}\right).$$

- $L_1 = \partial_x^2 - \frac{2}{x^2}, \quad L_2 = \partial_x^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}$

$$L_1^3 = L_2^2, \quad R(\lambda,\mu) = \lambda^3 - \mu^2.$$

Spectral curve

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 : R(\lambda, \mu) = 0\}.$$

If $L_1\psi = \lambda\psi$ and $L_2\psi = \mu\psi$, then $(\lambda, \mu) \in \Gamma, \psi = \psi(x, P), \; P = (z, w)$.

rank of L_1 and L_2 is

$$l = \dim\{\psi : L_1\psi = \lambda\psi, \; L_2\psi = \mu\psi\}.$$

Commuting differential operators of rank one

Baker–Akhiezer function $\psi(x, P), P \in \Gamma$
Spectral data

$$\{\Gamma, q, k^{-1}, \gamma_1, \dots, \gamma_g\}$$

Γ is a Riemann surface,
k^{-1} is a local parameter near $q \in \Gamma$,
$\gamma_1, \dots \gamma_g \in \Gamma$.
The Baker–Akhiezer function has the properties:

- $\psi = e^{kx}\left(1 + \frac{f(x)}{k} + \dots\right)$
- on $\Gamma \backslash q$ the BA-function ψ is meromorphic with the poles in $\gamma_1, \dots, \gamma_g$

Commuting differential operators of rank one

Let f(P) be a meromorphic function on Γ with a unique pole in q of order n

$$f = k^n + c_{n-1}k^{n-1} + \dots + c_0 + \frac{c_{-1}}{k} + \dots$$

$$\partial_x^n \psi + u_{n-1}(x)\partial_x^{n-1}\psi + \dots + u_0(x)\psi = f\psi + e^{kx}\left(O\left(\frac{1}{k}\right)\right).$$

From the uniqueness of BA-function it follows that

$$L_1\psi(x,P) = f(p)\psi(x,P).$$

Let g(P) be a meromorphic function with unique pole in q of order m

$$L_2\psi(x,P) = g(P)\psi(x,P).$$

We have

$$(L_1L_2 - L_2L_1)\psi(x,P) = 0 \Rightarrow L_1L_2 = L_2L_1.$$

Example

$$\Gamma = \mathbb{C}P^1,\ q = \infty,\ k = z$$

Baker–Akhiezer function $\psi = e^{xz}$

$$f = z^n + c_{n-1}z^{n-1} + \cdots + c_0,$$

$$\partial_x^n \psi + c_{n-1}\partial_x^{n-1}\psi + \cdots + c_0\psi = f\psi.$$

Commuting differential operators of rank one

Example

$$\Gamma = \mathbb{C}/\{2\omega\mathbb{Z} + 2\omega'\mathbb{Z}\}, \; q = 0,$$

$$\psi = e^{-x\zeta(z)} \frac{\sigma(z + x + \gamma)}{\sigma(x + \gamma)\sigma(z + \gamma)},$$

$$L_2\psi = (\partial_x^2 - 2\wp(x + \gamma))\psi = \wp(z)\psi,$$

$$L_3\psi = \left(\partial_x^3 - 3\wp(x + \gamma)\partial_x - \frac{3}{2}\wp'(x + \gamma)\right)\psi = \frac{1}{2}\wp'(z)\psi,$$

$$L_3^2 = L_2^3 - \frac{g_2}{4}L_2 - \frac{g_3}{4},$$

Commuting differential operators of rank one

Under the degeneration $g_2, g_3 \to 0$ we get the caspidal spectral curve. Under this degeneration the functions $\sigma(z), \zeta(z), \wp(z)$ become

$$\hat{\sigma}(z) = z, \qquad \hat{\zeta}(z) = \frac{1}{z}, \qquad \hat{\wp}(z) = \frac{1}{z^2}.$$

We get commuting differential operators with rational coefficients

$$\hat{\psi}(x,z) = e^{-\frac{x}{z}} \frac{z + x + \gamma}{(x+\gamma)(z+\gamma)},$$

$$\hat{L}_2 \hat{\psi} = \left(\partial_x^2 - \frac{2}{(x+\gamma)^2} \right) \hat{\psi} = \frac{1}{z^2} \hat{\psi},$$

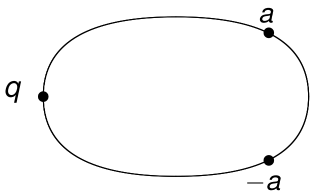
$$\hat{L}_3 \hat{\psi} = \left(\partial_x^3 - \frac{3}{(x+2)^2} \partial_x + \frac{3}{(x+\gamma)^3} \right) \hat{\psi} = -\frac{1}{z^3} \hat{\psi},$$

$$\hat{L}_2^3 = \hat{L}_3^2.$$

Commuting differential operators of rank one

Example

$$\Gamma = \mathbb{C}P^1/\{a \sim -a\},\ q = \infty,\ g_a = 1,\ k = z$$



$$\psi = e^{xz}\left(1 + \frac{\xi(x)}{z-\gamma}\right),$$

$$\psi(x,a) = \psi(x,-a)$$

$$\xi(x) = \frac{(\gamma^2 - a^2)\sinh(ax)}{a\cosh(ax) + \gamma\sinh(x)}.$$

Commuting differential operators of rank one

The functions $f(z) = z^2, g(z) = z^3 - a^2 z$ are rational functions on Γ with the poles of order 2 and 3 at q. Thus we have

$$L(f)\psi = (\partial_x^2 + u(x))\psi = z^2\psi,$$

$$L(g)\psi = \left(\partial_x^3 + \left(\frac{3}{2}u(x) - a^2\right)\partial_x + \frac{3}{4}u'(x)\right)\psi = (z^3 - a^2 z)\psi,$$

$$u(x) = \frac{2a^2(a^2 - \gamma^2)}{(a\cosh(ax) + \gamma\sinh(ax))^2}.$$

The Burchnall–Chaundy polynomial of L_1, L_2 is

$$R(\lambda, \mu) = \lambda^2 - \mu(\mu - a^2)^2.$$

Rank $l > 1$
Spectral data (Krichever)

$$\{\Gamma, q, k^{-1}, \gamma_1, \dots, \gamma_{lg}, \alpha_1, \dots, \alpha_{lg}\}$$

$\alpha_i = (\alpha_{1i}, \dots, \alpha_{il-1})$ — vector
(γ, α) — Turin parameters define stable (in the sense of Mumford) vector bundle of rank l degree lg on Γ with holomorphic sections $\eta_1, \dots, \eta_l$

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{ij}\eta_j(\gamma_i).$$

Vector Baker–Akhiezer function

$\psi(x,P)=(\psi_0(x,P),\dots,\psi_{l-1}(x,P))$:

1. $\psi(x,P)=\left(\sum_{s=0}^{\infty}\xi_s(x)k^{-s}\right)\Psi_0(x,P)$,

$\xi_0=(1,0,\dots,0)$, $\frac{d}{dx}\Psi_0=A\Psi_0$,

$$A=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1\\ k+u_0(x) & u_1(x) & u_2(x) & \dots & u_{l-1}(x) & 0 \end{pmatrix}$$

2. on $\Gamma-\{q\}$ ψ is meromorphic with the simple poles in $\gamma_1,\dots,\gamma_{lg}$

3. $\mathrm{Res}_{\gamma_i}\psi_j=\alpha_{ij}\mathrm{Res}_{\gamma_i}\psi_{l-1}$.

If f(P) is meromorphic function with the pole in q of order n, then there exist $L(f)$ such that

$$L(f)\psi(x,P)=f(P)\psi(x,P),\ \mathrm{ord}L(f)=ln.$$

Method of Turin parameters deformation

$$\frac{d^l}{dx^l}\psi_j = \chi_{l-1}\frac{d^{l-1}}{dx^{l-1}}\psi_j + \cdots + \chi_0\psi_j$$

χ_s — meromorphic on Γ, χ_s has lg simple poles $P_1(x), \ldots, P_{lg}(x)$. In the neighbourhood of q the functions χ_s have the form

$$\chi_0(x,P) = k + g_0(x) + O(k^{-1}),$$

$$\chi_j(x,P) = g_j(x) + O(k^{-1}), \quad j < l-1,$$

$$\chi_{l-1}(x,P) = O(k^{-1}).$$

Higher rank commuting differential operators

At the point $P_i(x)$

$$\chi_j = \frac{c_{ij}(x)}{k-\gamma_i(x)} + d_{ij}(x) + O(k-\gamma_i(x)).$$

Theorem (Krichever)

Parameters $\gamma_i(x), \alpha_{ij}(x) = \frac{c_{ij}(x)}{c_{i,l-1}(x)}$, *and* $d_{ij}(x), 0 \leq j \leq l-2, 1 \leq i \leq lg$ *satisfy the equation*

$$c_{i,l-1}(x) = -\gamma_i'(x),$$

$$d_{i0}(x) = \alpha_{i0}(x)\alpha_{i,l-2}(x) + \alpha_{i0}(x)d_{i,l-1}(x) - \alpha_{i0}'(x),$$

$$d_{ij}(x) = \alpha_{ij}(x)\alpha_{i,l-2}(x) - \alpha_{i,j-1}(x) + \alpha_{ij}(x)d_{i,l-1}(x) - \alpha_{ij}'(x), j \geq 1.$$

Krichever, Novikov: $g = 1,\ l = 2$ $\Gamma : \mu^2 = P_3(\lambda) = 4\lambda^3 + g_2\lambda + g_3$

$$L_{KN} = (\partial_x^2+u)^2+2c_x(\wp(\gamma_2)-\wp(\gamma_1))\partial_x+(c_x(\wp(\gamma_2)-\wp(\gamma_1)))_x-\wp(\gamma_2)-\wp(\gamma_1),$$

$$\gamma_1(x) = \gamma_0 + c(x),\ \gamma_2(x) = \gamma_0 - c(x),$$

$$u = -\frac{1}{4c_x^2}+\frac{1}{2}\frac{c_{xx}^2}{c_x^2}+2\Phi(\gamma_1,\gamma_2)c_x-\frac{c_{xxx}}{2c_x}+c_x^2(\Phi_c(\gamma_0+c,\gamma_0-c)-\Phi^2(\gamma_1,\gamma_2)),$$

$$\Phi(\gamma_1,\gamma_2) = \zeta(\gamma_2-\gamma_1)+\zeta(\gamma_1)-\zeta(\gamma_2).$$

Operator L_2 can be found from the equation $\tilde{L}_{KN}^2 = P_3(L_{KN})$.

Higher rank commuting differential operators

Let $A_1 = \mathbb{C}\langle p, q : [p, q] = 1\rangle$ be the Weyl algebra.

Theorem (Dixmier)

Two elements of A_1

$$X = (p^3 + q^2 + h)^2 + 2p,$$

$$Y = (p^3 + q^2 + h)^3 + \frac{3}{2}\left(p(p^3 + q^2 + h) + (p^3 + q^2 + h)p\right), \; h \in \mathbb{C}$$

commute and satisfy the equation $Y^2 = X^3 - h$.

If $p = x$, $q = -\partial_x$ ($[x, -\partial_x] = 1$), then we get operators of rank two

$$L_D = (\partial_x^2 + x^3 + h)^2 + 2x,$$

$$\tilde{L}_D = \left(\partial_x^2 + x^3 + h\right)^3 + \frac{3}{2}\left(x\left(\partial_x^2 + x^3 + h\right) + \left(\partial_x^2 + x^3 + h\right)x\right).$$

Operator L_D coincides with L_{KN} for some $c(x)$. Then, a natural question is how to obtain L_D from L_{KN} (Gelfand's problem).

Higher rank commuting differential operators

Theorem (Grinevich)

Operators L_{KN} and $\tilde{L}_{KN}$ have rational coefficients if and only if

$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{P_3(t)}},$$

where $q(t)$ is a rational function.
If $\gamma_0 = 0$, and $q(x) = x$, we have the Dixmier operators.

Theorem (Grinevich, Novikov)

Operator L_{KN} is formally self-adjoint if and only if $\wp(\gamma_1) = \wp(\gamma_2)$.

Mokhov: $g = 1, \ l = 3$

Rank $l = 2,\ g > 1$: self-adjoint case

$$\Gamma : w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \dots + c_0, \quad q = \infty$$

$$L_4\psi = z\psi, \quad L_{4g+2}\psi = w\psi, \ (L_{4g+2})^2 = F_g(L_4),$$

$$\sigma : \Gamma \to \Gamma, \ \sigma(z, w) = (z, -w).$$

We have

$$\psi''(x, P) = \chi_1(x, P)\psi'(x, P) + \chi_0(x, P)\psi(x, P), \ P = (z, w) \in \Gamma,$$

where $\psi = (\psi_1, \psi_2)$ is a Baker–Akhiezer function.

Theorem (M.)

The operator L_4 is self-adjoint if and only if

$$\chi_1(x,P)=\chi_1(x,\sigma(P)).$$

At $g=1$ the Theorem was proved by Grinevich and Novikov.
Let us assume that the operator L_4 is self-adjoint

$$L_4=(\partial_x^2+V(x))^2+W(x),$$

then the functions χ_0,χ_1 have simple poles at some points

$$\left(\gamma_i(x),\pm\sqrt{F_g(\gamma_i(x))}\right)\in\Gamma,\ \ 1\leq i\leq g.$$

Theorem (M.)

If operator L_4 is self-adjoint, then

$$\chi_0 = -\frac{1}{2}\frac{Q''}{Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q'}{Q},$$

where

$$Q = (z - \gamma_1(x)) \dots (z - \gamma_g(x)).$$

Function Q satisfies the equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q^{(3)}$$

$$+2Q(2V'Q' + 4VQ'' + Q^{(4)}),$$

where $Q', Q'', Q^{(k)}$ mean $\partial_x Q, \partial_x^2 Q, \partial_x^k Q$.

Higher rank commuting differential operators

Corollary (M.)

The function Q satisfies the linear equation

$$\mathcal{L}_5 Q = \left(\partial_x^5 + 2\langle V, \partial_x^3\rangle + 2\langle z - W - V'', \partial_x\rangle\right) Q = 0,$$

where $\langle A, B\rangle = AB + BA$. *Potentials* V, W *have the form*

$$V = \left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2}\right)|_{z=\gamma_j},$$

$$W = -2(\gamma_1 + \cdots + \gamma_g) - c_{2g}.$$

The functions $\gamma_1(x), \ldots, \gamma_g(x)$ *satisfy the equations*

$$\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2}|_{z=\gamma_j} = \frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2}|_{z=\gamma_k}.$$

Higher rank commuting differential operators

Theorem (M.)

The operator

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \qquad \alpha_3 \neq 0$$

commutes with a differential operator $L_{4g+2}^\sharp$ of order $4g+2$. The operators $L_4^\sharp$, $L_{4g+2}^\sharp$ are operators of rank two. For generic values of parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ the spectral curve is a nonsingular hyperelliptic curve of genus g.

If $g = 1$, $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, then the operators $L_4^\sharp$, $L_{4g+2}^\sharp$ coincide with the Dixmier operators

Higher rank commuting differential operators

At $g = 1$ we have

$$V = \frac{-16F_1(\gamma) + W''^2 - 2W'W'''}{4W'^2}, \quad \gamma = \frac{-c_2 - W}{2},$$

$$L_4 = (\partial_x^2 + V(x))^2 + W(x).$$

Higher rank commuting differential operators

Let L_2 be a finite-gap Schrödinger operator

$$L_2\psi = (-\partial_x^2 + u(x))\psi = z\psi.$$

$\Gamma : w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \dots + c_0,\ \ q = \infty,\ L_2 L_{2g+1} = L_{2g+1} L_g.$

The BA function $\psi = \psi(x, P)$, $P = (z, w) \in \Gamma$ has g zeros

$$(\gamma_j(x), w(\gamma_j(x))) \in \Gamma.$$

$Q = (z - \gamma_1(x)) \dots (z - \gamma_g(x))$ satisfies the equations

$$4F_g(z) = 4(z - u)Q^2 - (Q')^2 + QQ'',$$

$$\mathcal{L}_3 Q = \left(\partial_x^3 + 2\langle z - u, \partial_x\rangle\right) Q = 0,$$

$$u = -2(\gamma_1 + \dots + \gamma_g) - c_{2g}.$$

Functions $\gamma_1, \dots, \gamma_g$ satisfy Dubrovin's equations

$$\gamma_j' = \pm\frac{2i\sqrt{F_g(\gamma_j)}}{\prod_{k \neq j}(\gamma_k - \gamma_j)}.$$

Evolution equations

$$[L_4, \partial_{t_n} - A_n] = 0,$$

$$L_4 = (\partial_x^2 + V(x, t_n))^2 + W(x, t_n),$$

$$-A_n^* = A_n = \partial_x^{2n+1} + \dots$$

$$A_3 = \partial_x^3 + \frac{3}{2} V(x, t)\partial_x + \frac{3}{2} V^{'}(x, t),$$

$$V_{t_1} = \frac{1}{4}(6VV^{'} + 6W^{'} + V^{'''}), \quad W_{t_1} = \frac{1}{2}(-3VW^{'} - W^{'''}),$$

Drinfeld and Sokolov found solutions of rank 1

$$[L_4, L_{2g+1}] = 0.$$

Evolution equations

Solutions of rank two

$$[L_4, \partial_{t_n} - A_n] = 0, \quad [L_4, L_{4g+2}] = 0$$

Theorem (Davletshina, M.)

$$Q_{t_1} = \frac{1}{2}(-3VQ' - Q'''),$$

$$Q_{t_2} = \frac{1}{8}(-4QW' + 2V'Q'' + Q'(8z - 5V^2 + 2W - V'') - 2VQ''').$$

These equations give symmetries of

$$4F_g(z) = 4Q^2(z - W) - 4VQ'^2 + Q''^2 - 2Q'Q''' + 2Q(2Q'V' + 4VQ'' + Q^4).$$

At $g = 1, n = 1$ we have the Krichever–Novikov equation

$$W_{t_1} = \frac{48F_1(\gamma) - W''^2 + 2W'W'''}{8W'}, \quad \gamma = \frac{-c_2 - W}{2}$$

Open problems

Krichever–Novikov operators up to the conjugations are self-adjoint operators.

Theorem (Latham, Previato)

$$L_4 = (\partial_x^2 + V(x))^2 + W(x), \; g = 1$$

$$L_4 - z_0 = A_2 T, \qquad L_6 - w_0 = A_4 T, \qquad T = \partial_x^2 - \chi_1 \partial_x - \chi_0.$$

We have

$$L_{KN} = TA_2 = T(L_4 - z_0)T^{-1}, \qquad \tilde{L}_{KN} = TA_4 = T(L_6 - w_0)T^{-1}$$

To prove an analog of the Theorem for $g > 1$.

Open problems

Kadomtsev–Petviashvili equation

$$\frac{3}{4}U_{yy} = \partial_x\left(U_t + \frac{3}{2}UU_x - \frac{1}{4}U_{xxx}\right)$$

is equivalent to

$$[\partial_y - M, \partial_t - A] = 0,$$

where

$$M = \partial_x^2 - U(x,y,t), \quad A = \partial_x^3 - \frac{3}{2}U\partial_x + S(x,y,t),$$

$$S_x = -\frac{3}{4}U_y - \frac{3}{4}U_{xx}, \quad S_y = -U_t - \frac{3}{4}U_{xy} + \frac{U_{xxx}}{4} - \frac{3}{2}UU_x$$

Krichever found rank one solutions of KP

$$U = 2\partial_x^2 \log\theta(V_1x + V_2y + V_3t + V_4, \Omega).$$

Shiota proved the Novikov conjecture.

Open problems

Rank two, $g = 1$ solutions of KP were found by Krichever and Novikov

$$[L_{KN}, \partial_y - M] = 0,$$

$$L_{KN} = (\partial_x^2 - U)^2 + f_1 \partial_x + \partial_x f_1 + f_0,$$

$$U = -V - \frac{-2W'^2 + 2(c_2 + W + 2z(y))}{(c_2 + W + 2z(y))^2},$$

$$V = \frac{-16F_1(\gamma) + W''^2 - 2W' W'''}{4W'^2}, \quad \gamma = \frac{-c_2 - W}{2},$$

$W = W(x, t)$ satisfies the Krichever–Novikov equation

$$W_t = \frac{48F_1(\gamma) - W''^2 + 2W' W'''}{8W'},$$

$z(y)$ satisfies the following equation

$$(z')^2 = 4F_1(z).$$

To find rank two solutions of KP ($g > 1$).

Open problems

The group of automorphisms of the first Weyl algebra $Aut(A_1)$ acts on the moduli spaces of operators with polynomial coefficients. For example, with the help of the automorphism

$$\varphi_1(x) = \alpha x + \beta \partial_x, \qquad \varphi_1(\partial_x) = \gamma x + \delta \partial_x, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2$$

one can get from $L_4^\sharp, L_{4g+2}^\sharp$ the operators of rank 3

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \qquad \alpha_3 \neq 0.$$

Another example of automorphisms are

$$\varphi_2(x) = x + P_1(\partial_x), \qquad \varphi_2(\partial_x) = \partial_x,$$

$$\varphi_3(x) = x, \qquad \varphi_3(\partial_x) = \partial_x + P_2(x),$$

where P_1, P_2 are polynomials. Dixmier proved that $Aut(A_1)$ is generated by φ_i. It would be very interesting to understand how $Aut(A_1)$ acts on the spectral data.

Open problems

The equation

$$Y^2 = X^{2g+1} + c_{2g}X^{2g} + \cdots + c_0$$

has nonconstant solutions $X = L_4^\sharp$, $Y = L_{4g+2}^\sharp \in A_1$ for some c_i.
It is easy to see that the group $Aut(A_1)$ preserves the space of all such solutions, i.e. if (X, Y) is a solution to the polynomial equation above, with $X, Y \in A_1$, then $(\varphi(X), \varphi(Y))$ is also a solution for any $\varphi \in Aut(A_1)$. Then, a natural question is to describe the orbits of $Aut(A_1)$ in the space of solutions under the action of $Aut(A_1)$.
Yu. Berest has proposed the following conjecture: If $g > 1$, then there are only finitely many such orbits, i.e. the equation $f(X, Y) = \sum_{i,j=0}^k \alpha_{ij}X^iY^j = 0$ with generic $\alpha_{ij} \in \mathbb{C}$ has at most finitely many solutions in A_1 up to the action of $Aut(A_1)$.

Open problems

The Dixmier conjecture:

$$End(A_1) = Aut(A_1).$$

If one describe all orbits of $Aut(A_1)$ in the space of solutions for the equation $f(X, Y) = 0$, then this gives a chance to compare $End(A_1)$ and $Aut(A_1)$. For example, if there is only one orbit, then $End(A_1) = Aut(A_1)$. For this reason it is important to find all solutions $X, Y \in A_1$ for one concrete equation and to study the action of $Aut(A_1)$. For example, one can take the simplest equation $Y^2 = X^3 + 1$.