Sparse semigroups: structure and bounds

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Abstract

Let $\mathbb{N}$ be the set of non-negative integers and $H = \{0 = n_1 < n_2 < \ldots\} \subseteq \mathbb{N}$ be a *numerical (additive) semigroup of finite genus* g. , i.e., the complement $\mathbb{N}\setminus H$ has g elements called *gaps*, $\mathrm{Gaps}(H) = \{\ell_1, \ldots, \ell_g\}$. For any numerical semigroup H, we define the parameter $m := n_2 - 1$. Weierstrass semigroups on algebraic curves are examples of numerical semigroups. It is known that a numerical semigroup may not always be realized as Weierstrass semigroup of a point on an algebraic curve. Thus, we can look at these semigroups from a purely algebraic point of view.

A semigroup $H = \{0 = n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ is said to be *sparse* (as defined by Munuera, Torres and Villanueva) if its set of gaps $\mathrm{Gaps}(H) = \{\ell_1, \ldots, \ell_g\}$ satisfies $\ell_i - \ell_{i-1} \leq 2, \quad \text{for} \quad i = 2, \ldots, g$ (equivalently, $n_{i+1} - n_i \geq 2,\ i = 1, \ldots, c - g$, where $c = \ell_g + 1$). Thus, each pair of gaps in a sparse semigroup differ by 1 or 2. We present a count of how many gaps are consecutive and how many differ by 2.

It is well known (Oliveira) that $\ell_g \leq 2g - 1$. The cases of even or odd ℓ_g have to be handled separately and the techniques are considerably different.

We present a partial classification of sparse semigroups having odd largest gap ℓ_g.

In case ℓ_g is even, we write $\ell_g = 2g - 2k$. Munuera, Torres and Villanueva presented an upper bound for the genus of such semigroups:

Theorem 1 *Let H be a sparse semigroup of genus g with $\ell_g = 2g - 2k$. If $g \geq 4k - 1$, then $g \leq 6k - n_1$.*

We show that there are no sparse semigroups having genus $g > 4k - 1$ and even largest gap $\ell_g = 2g - 2k$, yielding a new upper bound for their genus.

AMS Classification: 20M14.