Sparse semigroups: structure and bounds

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Abstract

SPARSE semigroups were defined and firstly explored in [13]. This work develops further the theme, presenting results on their structure. We also obtain a sharp quota for the genus of sparse semigroups having even largest gap. In addition, we characterize completely some instances of sparse semigroups having odd largest gap.

1. Introduction

LET $\mathbb{N}$ be the set of non-negative integers and $\mathcal{H} = \{0 = n_0 < n_1 < \ldots\} \subseteq \mathbb{N}$ be a *numerical (additive) semigroup of finite genus* g, i.e., the complement $\mathbb{N} \setminus \mathcal{H}$ has g elements called *gaps*, $\mathrm{Gaps}(\mathcal{H}) = \{\ell_1, \ldots, \ell_g\}$. This is the same as saying that the least common divisor among the generators of $\mathcal{H}$ is 1. The elements of $\mathcal{H}$ are refered to as *nongaps*. The largest gap ℓ_g is called the *Frobenius number* of $\mathcal{H}$. The smallest nonzero element n_1 in $\mathcal{H}$ is said to be its *multiplicity*.

Weierstrass semigroups in algebraic curves are examples of numerical semigroups. Although some motivation for studying numerical semigroups comes from the theory of curves, the techniques we use rely on the semigroup structure alone, so problems in this field can be stated in a more general context. Furthermore, it is known that a numerical semigroup cannot always be realized as a Weierstrass semigroup of a point on an algebraic curve [14, 9]; so numerical semigroups are interesting in their own right.

Among numerical semigroups, Arf semigroups have been attracting a lot of attention. Several authors have studied one-dimensional analytically unramified domains via their valuation semigroups (for instance, [3, 7, 8, 10, 12, 16, 19]). One of the properties studied via this approach for those rings is the Arf property. From [1], Lipman in [11] introduces and motivates the study of Arf rings, whose characterization in terms of their semigroups of values gives rise to the notion of Arf semigroup. In [2], a connection is established between the Arf property of a one-dimensional analytically irreducible domain and the Arf property of its semigroup of values. The relationship between the Pythagorean property of a real curve germ and the Arf property of its value numerical semigroup was studied in [6, 15]. Besides local algebra theory, the study of Arf semigroups has been carried out in the context of coding theory [4, 5].

In [13], the authors remarked that the elements in an Arf semigroup satisfy a condition of minimum "spacing" (so the gaps satisfy a condition of maximum spacing) and thus proposed the definition of sparse semigroups as being semigroups that satisfy this condition. In this work, we take the study of sparse semigroups further as we analyze their structure more deeply.

2. Sparse semigroups

FOR any numerical semigroup with Frobenius number ℓ_g, it is well known that $\ell_g \leq 2g - 1$ [14, Theorem 1.1]. We may, thus, define the parameter $K := 2g - \ell_g \geq 1$.

Theorem 2.1 *Let $\mathcal{H}$ be any numerical semigroup.*

1. *For every gap ℓ_i in $\mathcal{H}$, there exists a well defined injective reflection function $R_i : \mathcal{H} \cap \{1, 2, \ldots, \ell_i\} \longrightarrow \mathrm{Gaps}(\mathcal{H})$, $h \mapsto \ell_i - h$.*
2. *For every gap ℓ_i in $\mathcal{H}$, if d divides ℓ_i, then $d \in \mathrm{Gaps}(\mathcal{H})$.*

We say a numerical semigroup $\mathcal{H} = \{0 = n_0 < n_1 < \ldots\}$ is an *Arf semigroup* if $n_i + n_j - n_k \in \mathcal{H}$, for $i \geq j \geq k$. Several alternative characterizations of Arf semigroups can be given ([2, Theorem I.3.4] presents fifteen equivalent ones). Any Arf semigroup satisfies a condition of minimum "spacing" among its elements: its set of gaps $\mathrm{Gaps}(\mathcal{H}) = \{\ell_1, \ldots, \ell_g\}$ satisfies $\ell_i - \ell_{i-1} \leq 2$, for $i = 2, \ldots, g$ (equivalently, $n_{i+1} - n_i \geq 2,\ i = 1, \ldots, c - g$) [13, Corollary 2.2]. A semigroup $\mathcal{H}$ is said to be *sparse* if it has the previous property. There are sparse semigroups that are not Arf [13, Example 2.3], which is a motivation for studying this class of semigroups.

Since sparse semigroups are the ones where subsequent gaps are either consecutive or spaced by 2, it is only natural to count how many pairs of subsequent gaps are in either situation. Given a sparse semigroup $\mathcal{H}$, consider the sets and their cardinalities:

$$\mathcal{D} := \{i \; ; \; \ell_{i+1} - \ell_i = 2\} \text{ ("double leaps")}, \; D := \#\mathcal{D},$$

$$\mathcal{S} := \{i \; ; \; \ell_{i+1} - \ell_i = 1\} \text{ ("single leaps")}, \; S := \#\mathcal{S}.$$

Proposition 2.2 *Let $\mathcal{H}$ be a sparse semigroup of genus g. Then:*

1. $D + S = g - 1$.
2. $D = g - K$.
3. $S = K - 1$.

3. Structure and Bounds

NEXT proposition gives us a little bit more information on the structure of sparse semigroups. It tells us that, if $g \geq 2K - 1$, then the last few gaps occur every two integers.

Proposition 3.1 *Let $\mathcal{H}$ be a sparse semigroup of genus g with Frobenius number $\ell_g = 2g - K$. If $g \geq 2K - 1$, then $\ell_{i+1} - \ell_i = 2$, for every $i = 2K - 2, \ldots, g - 1$.*

Upon researching sparse semigroups, it became clear to us that those having genus $g = 2K - 1$ and Frobenius number $\ell_g = 2g - K = 3K - 2$ are quite special. In fact, the lemma below suggests that they are "limit cases" and "maximal" in some sense.

Lemma 3.2 *Let $\mathcal{H}$ be a sparse semigroup of genus $g = 2K + j,\ j \geq 0$, with Frobenius number $\ell_g = 2g - K$. Then there is a sparse semigroup $\widetilde{\mathcal{H}}$ of genus $\widetilde{g} = 2K - 1$ and Frobenius number $\ell_{\widetilde{g}} = 2\widetilde{g} - K = 3K - 2$ such $\mathcal{H}$ is a subsemigroup of $\widetilde{\mathcal{H}}$.*

We investigate sparse semigroups that have as many single leaps as double leaps, i. e., $S = D$. These semigroups are in some sense "limit cases", and the primary motivation for us to study them came from an upper bound for the genus of sparse semigroups with even Frobenius number by Munuera, Torres and Villanueva [13]. Recall that, in our notation, $\ell_g = 2g - K$; if ℓ_g is even, so is K and we may write $K = 2k$, $\ell_g = 2g - 2k$.

Theorem 3.3 *[13, Theorem 3.1] Let $\mathcal{H}$ be a sparse semigroup of genus g with Frobenius number $\ell_g = 2g - 2k$. If $g \geq 4k - 1$, then $g \leq 6k - n_1$.*

Observe that the hypothesis of this theorem is the same of Proposition 3.1. It is only natural for one to ask about the existence of sparse semigroups of genus $g \geq 4k - 1$ and Frobenius number $\ell_g = 2g - 2k$, specially since $6k - n_1$ is such a "loose" upper bound for g. A preliminary analysis of examples using Maple software suggests that there are no such semigroups if $g > 4k - 1$, which would yield a new upper bound for their genus (a tight one, since there exist such semigroups having $g = 4k - 1$). This fact together with Lemma 3.2 reinforces the idea that sparse semigroups with even Frobenius number $\ell_g = 2g - 2k$ and genus $g = 4k - 1$ are special. Notice, in particular, that Proposition 2.2 tells us that such semigroups have $S = D$, i.e., the same number of single and double leaps. In fact, the converse is also true:

Lemma 3.4 *Let $\mathcal{H}$ be a sparse semigroup with Frobenius number $\ell_g = 2g - K$ and as many single as double leaps, i.e., $S = D$. Then $g = 2K - 1$ and is an odd number.*

Lemma 3.5 *Let $\mathcal{H}$ be a sparse semigroup with Frobenius number $\ell_g = 2g - K$ and as many single as double leaps, i.e., $S = D$. Then $\#(\mathcal{H} \cap \{1, \ldots, \ell_g\}) = S = D$.*

Let us now see that there is precisely one sparse semigroup having genus $g = 4k - 1$ and even Frobenius number $\ell_g = 2g - 2k$.

Lemma 3.6 *Let $\mathcal{H}$ be a sparse semigroup of genus $g = 4k - 1$ and Frobenius number $\ell_g = 2g - 2k = 6k - 2$. Then $3 \in \mathcal{H}$ if and only if $6k - 5 \notin \mathcal{H}$.*

Theorem 3.7 *Let $\mathcal{H}$ be a sparse semigroup of genus $g = 4k - 1$ with Frobenius number $\ell_g = 2g - 2k$. Then $3 \in \mathcal{H}$, i.e.,*

$$\mathcal{H} = \langle 3, 6k - 1, 6k + 1 \rangle = 3\mathbb{N} \cup \{n \in \mathbb{N} \; ; \; n \geq 6k - 1\}.$$

Corollary 3.8 *If $\mathcal{H}$ is a sparse numerical semigroup with genus $g = 4k - 1$ and even Frobenius number $\ell_g = 2g - 2k$ and as many single as double leaps, i.e., $S = D$, then $\mathcal{H}$ is an Arf semigroup. Moreover, $\mathcal{H}$ is a Weierstrass semigroup.*

We now have a tight quota for the genus of a sparse semigroup with even Frobenius number, greatly improving that by Munuera, Torres and Villanueva (Theorem 3.9):

Theorem 3.9 *Let $\mathcal{H}$ be a sparse semigroup of genus g with Frobenius number $\ell_g = 2g - 2k$. Then $g \leq 4k - 1$.*

Let us now analyze some constraints on the multiplicity of a sparse semigroup having odd Frobenius number.

Theorem 3.10 *Let $\mathcal{H}$ be a sparse semigroup of genus $g = 4k+1$ and odd Frobenius number $\ell_g = 2g-(2k+1) = 6k+1$. If the multiplicity n_1 of $\mathcal{H}$ is even, then every nongap smaller then ℓ_g is even.*

Theorem 3.11 *Let $\mathcal{H}$ be a sparse semigroup of genus $g = 4k+1$ and odd Frobenius number $\ell_g = 2g-(2k+1) = 6k+1$. If the multiplicity n_1 of $\mathcal{H}$ is odd, then $\mathcal{H}$ is one of the following:*

1. $\mathcal{H} = 3\mathbb{N} \cup \{n \in \mathbb{N} \; ; \; n \geq 6k + 2\}$;
2. $\mathcal{H} = \{0\} \cup \{2j + 1; j \in \mathbb{N}, k \leq j \leq 2k - 1\} \cup \{2j; j \in \mathbb{N}, 2k + 1 \leq j \leq 3k\} \cup \{n \in \mathbb{N} \; ; \; n \geq 6k + 2\}$.

Theorem 3.12 *Let $\mathcal{H}$ be an sparse semigroup of genus g with $\ell_g = 2g - (2k + 1)$. Then $g \leq 4k + 1$, or all nongaps smaller than ℓ_g are even.*

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