Stochastic calculus with respect to the fractional Brownian motion

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Outline

1. Fractional Brownian motion
2. Stochastic calculus. Riemann sums approach
3. Stochastic heat equation driven by a fractional noise

Stochastic Processes

- A stochastic process $X = \{X_t, t \geq 0\}$ is a family of random variables
$$X_t : \Omega \to \mathbb{R}$$
defined on a probability space $(\Omega, \mathcal{F}, P)$
- X is called *Gaussian* if for all $0 \leq t_1 < t_2 < \cdots < t_n$ the probability law of $(X_{t_1}, \ldots, X_{t_n})$ on $\mathbb{R}^n$ is normal
- The law of a Gaussian process is determined by the mean function $\mathbb{E}(X_t)$ and the covariance function
$$\mathrm{Cov}(X_t, X_s) = \mathbb{E}((X_t - \mathbb{E}(X_t))(X_s - \mathbb{E}(X_s)))$$

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Brownian Motion

The Brownian motion (or Wiener process) is a Gaussian process $W = \{W_t, t \geq 0\}$ with zero mean and covariance

$$\mathbb{E}(W_s W_t) = \min(s, t)$$

- $\mathbb{E}(W_t - W_s)^2 = |t - s|$
- W has independent increments
- The formal derivative $\frac{dW_t}{dt}$ is used as input noise in dynamical systems
- The stochastic calculus developed by Itô in the 40's permits to formulate and solve stochastic differential equations $dX_t = b(X_t)dt + \sigma(X_t)dW_t$

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Applications in hydrology, telecommunications, queueing and mathematical finance require input noises without independent increments and possessing:

- Stationary and correlated increments
- Irregular trajectories $t \to X_t(\omega)$

Fractional Brownian Motion

The fractional Brownian motion (fBm) $B^H = \{B_t^H, t \geq 0\}$ is a zero mean Gaussian process with covariance

$$\mathbb{E}(B_s^H B_t^H) = R_H(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

$H \in (0,1)$ is called the Hurst parameter

- $\mathbb{E}(B_t^H - B_s^H)^2 = |t-s|^{2H}$
- For any $\gamma < H$, with probability one, the trajectories $t \to B_t^H(\omega)$ are Hölder continuous of order γ:

$$|B_t^H(\omega) - B_s^H(\omega)| \leq G_{\gamma,T}(\omega)|t-s|^{\gamma}, \quad s,t \in [0,T]$$

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Correlated increments

- For $H \neq \frac{1}{2}$,

$$\rho_H(n) = \mathbb{E}(B_1^H(B_{n+1}^H - B_n^H)) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right) \sim H(2H-1)n^{2H-2},$$

as $n \to \infty$

$\frac{1}{H}$*-variation* (Rogers '97)

- Fix $[0, T]$. Set $t_i = \frac{iT}{n}$ for $0 \leq i \leq n$ and define $\Delta B_{t_i}^H = B_{t_{i+1}}^H - B_{t_i}^H$. Then,

$$\sum_{i=0}^{n-1} |\Delta B_{t_i}^H|^{\frac{1}{H}} \xrightarrow{L^2(\Omega)} c_H T$$

Formally, $|dB_t^H|^{\frac{1}{H}} \sim c_H dt$

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Stochastic calculus

Problem: Give a meaning to the formal derivative $\frac{dB^H}{dt}$ (fractal noise), and define integrals with respect to dB^H_t of the form

$$\int_0^T \varphi_t dB^H_t$$

- For $H \neq \frac{1}{2}$, B^H is not a semimartingale and we cannot use Itô's stochastic calculus to define stochastic integrals

Integration of deterministic functions

- The integral of a step function $\varphi_t = \sum_{j=1}^m a_j \mathbf{1}_{(s_j, s_{j+1}]}(t) \in \mathcal{E}$, where $t \in [0, T]$, is defined by

$$\int_0^T \varphi_t dB_t^H = \sum_{j=1}^m a_j (B_{s_{j+1}}^H - B_{s_j}^H)$$

- Let $\mathcal{H}$ be the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = \mathbb{E}(B_t^H B_s^H),$$

- Linear isometry:

$$\mathcal{H} \longrightarrow L^2(\Omega, \mathcal{F}, P)$$
$$\varphi \longrightarrow \int_0^T \varphi_t dB_t^H$$

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Spaces of integrable functions

- If $H = \frac{1}{2}$, $\mathcal{H} = L^2([0,T])$ and (*Itô isometry*)

$$\mathbb{E}\left(\int_0^T \varphi_t dB_t^{\frac{1}{2}}\right)^2 = \int_0^T \varphi_t^2 dt$$

- If $H > \frac{1}{2}$, using that

$$\mathbb{E}\left(dB_t^H dB_s^H\right) = \frac{\partial R_H^2}{\partial s \partial t} dsdt = H(2H-1)|s-t|^{2H-2} dsdt$$

we obtain

$$\mathbb{E}\left(\int_0^T \varphi_t dB_t^H\right)^2 = \alpha_H \int_0^T \int_0^T \varphi_s \varphi_t |s-t|^{2H-2} dsdt,$$

where $\alpha_H = H(2H-1)$
The space $\mathcal{H}$ contains the set of functions φ such that

$$\int_0^T \int_0^T |\varphi_s||\varphi_t||s-t|^{2H-2} dsdt < \infty,$$

which includes $L^{\frac{1}{H}}([0,T[)$

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- If $H < \frac{1}{2}$, $\mathcal{H}$ is a space of functions:

$$\mathcal{H} = I_{T-}^{\frac{1}{2}-H}(L^2([0,T])),$$

where $I_{T-}^{\frac{1}{2}-H}$ is the fractional integral operator of order $\frac{1}{2} - H$

For any $\gamma > \frac{1}{2} - H$,

$$C^{\gamma}([0,T]) \subset \mathcal{H}$$

Integration of random processess

(I) *Case* $H > \frac{1}{2}$

- Fix $f \in C^2$, then $\int_0^T f'(B_t^H) dB_t^H$ exists as a path-wise Riemann-Stieltjes integral (Young '36), and

$$f(B_T^H) = f(0) + \int_0^T f'(B_t^H) dB_t^H$$

In fact, the trajectories $t \mapsto f'(B_t^H(\omega))$ and $t \mapsto B_t^H(\omega)$ are Hölder continuous of order larger than $\frac{1}{2}$

(II) *Case* $H = \frac{1}{2}$

- Forward Riemann sums converge to the Itô integral:

$$\sum_{i=0}^{n-1} f'(B^{\frac{1}{2}}_{t_i})\Delta B^{\frac{1}{2}}_{t_i} \xrightarrow{P} \int_0^T f'(B^{\frac{1}{2}}_t)\delta B^{\frac{1}{2}}_t$$

which satisfies the Itô formula

$$f(B^{\frac{1}{2}}_T) = f(0) + \int_0^T f'(B^{\frac{1}{2}}_t)\delta B^{\frac{1}{2}}_t + \frac{1}{2}\int_0^T f''(B^{\frac{1}{2}}_t)dt$$

- Symmetric Riemann sums converge to the Stratonovich integral

$$\int_0^T f'(B^{\frac{1}{2}}_t) \circ dB^{\frac{1}{2}}_t = f(B^{\frac{1}{2}}_T) - f(0)$$

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(III) *Case* $H < \frac{1}{2}$

- Forward Riemann sums diverge
- For $H > \frac{1}{4}$ midpoint Riemann sums converge:

$$S_n^{MP} = \sum_{i=0}^{n-1} f'\left(B^H_{t_i + \frac{T}{2n}}\right) \Delta B^H_{t_i} \xrightarrow{P} f(B^H_T) - f(0)$$

- For $H > \frac{1}{6}$ trapezoidal Riemann sums converge:

$$S_n^{TR} = \sum_{i=0}^{n-1} \frac{1}{2}\left(f'(B^H_{t_i}) + f(B^H_{t_{i+1}})\right) \Delta B^H_{t_i} \xrightarrow{P} f(B^H_T) - f(0),$$

- For $H = \frac{1}{4}$ and $H = \frac{1}{6}$ these sums diverge in $L^2(\Omega)$ for $f(x) = x^2$

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Convergence in law in the critical cases

Let W be a Brownian motion independent of B^H

- For $H = \frac{1}{4}$

$$S_n^{MP} \xrightarrow{\mathcal{L}} \int_0^T f'(B_t^{\frac{1}{4}}) * dB_t^{\frac{1}{4}} = f(B_T^{\frac{1}{4}}) - f(0) - \frac{\kappa_1}{2} \int_0^T f''(B_t^{\frac{1}{4}}) dW_t$$

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- $\kappa_1 = \sqrt{2 + \sum_{r=1}^{\infty} (-1)^r \rho_{\frac{1}{4}}(r)^2} \sim 1290$ and $\kappa_2 = \frac{1}{\sqrt{6}}$
- Proof is based on Taylor expansion and non central limit theorems for Skorohod integrals using techniques of Malliavin calculus (Burdzy, Swanson, Nourdin, Réveillac, Nualart, Harnett)

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Multidimensional case

Example:

$$\int_0^T B_t^{H,1} dB_t^{H,2},$$

where $B^{H,1}$ and $B^{H,2}$ are two independent fractional Brownian motions

- The critical value for any symmetric Riemann sum is $H = \frac{1}{4}$
- For $H = \frac{1}{4}$

$$(\log n)^{-\frac{1}{2}} \sum_{i=0}^{n-1} \frac{1}{2}\left(B_{t_i}^{\frac{1}{4},1} + B_{t_{i+1}}^{\frac{1}{4},1}\right) \Delta B_{t_i}^{\frac{1}{4},2} \xrightarrow{\mathcal{L}} \frac{1}{\sqrt{8}} W_T,$$

where W_T is $N(0, T)$

Heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \quad x \in \mathbb{R}^d$$

- Let $p_t(x) = (2\pi t)^{-\frac{d}{2}} \exp(-|x|^2/2t)$. The solution with initial condition u_0 is

$$u(t,x) = p_t * u_0(x) = \int_{\mathbb{R}^d} p_t(x-y)u_0(y)dy = \mathbb{E}\left(u_0(W_t^x)\right),$$

where $W_t^x = x + W_t$ is a d-dimensional Brownian motion starting from x

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Feynman-Kac formula

Heat equation with a potential $V(t,x)$;

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- Probabilistic representation of the solution:

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Stochastic heat equation

- We are interested in the case where the potential V is a fractional white noise:
$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^{d+1}B}{\partial t \partial x_1 \cdots \partial x_d}, \tag{1}$$
where $B = \{B_{t,x}, t \geq 0, x \in \mathbb{R}^d\}$ is a zero mean Gaussian random field with covariance
$$\mathbb{E}(B_{t,x}B_{s,y}) = R_{H_0}(s,t)\prod_{i=1}^{d} R_{H_i}(x_i, y_i),$$
- That is, B is a fractional Brownian sheet with Hurst parameter H_0 in the time variable and H_i, $1 \leq i \leq d$, in the space variables
- For $x, y \in \mathbb{R}$, $R_H(x,y) = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right)$

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Remarks

(i) This equation is formal because the potential $V = \frac{\partial^{d+1} B}{\partial t \partial x_1 \cdots \partial x_d}$ is not a function

(ii) We can write for a function V

$$\int_0^t V(r, W_{t-r}^x) dr = \int_0^t \int_{\mathbb{R}^d} \delta_0(W_{t-r}^x - y) V(r, y) dy dr,$$

where δ_0 is the Dirac delta function

(iii) Assume $H_i > \frac{1}{2}$ for all i. If $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, then

$$\mathbb{E}\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} \varphi_{t,x} dB_{t,x}\right)^2 = \alpha_{\mathbf{H}} \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(t,x)\varphi(s,y) \times |s-t|^{2H_0-2} \prod_{i=1}^{d} |x_i - y_i|^{2H_i-2} dsdtdxdy,$$

where $\alpha_{\mathbf{H}} = \prod_{i=0}^{d} H_i(2H_i - 1)$.

Let $\mathcal{H}_d$ be the class of functions or distributions such that this integral exists

Theorem (Hu-Nualart-Song '11)

Let W be a d-dimensional Brownian motion independent of B. Assume u_0 is bounded and

$$2H_0 + \sum_{i=1}^{d} H_i > d+1.$$

Then

$$u(t,x) = \mathbb{E}^W \left[u_0(W_t^x) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta_0(W_{t-r}^x - y) dB_{r,y} \right) \right]$$

is well defined and satisfies Equation (1), where $\mathbb{E}^W$ denotes the expectation with respect to W

Sketch of the proof

STEP 1

Set

$$\Phi(t,x) = \int_0^t \int_{\mathbb{R}^d} \delta_0(W_{t-r}^x - y) dB_{r,y}$$

We claim that $(r,y) \mapsto \delta_0(W_{t-r}^x - y)\mathbf{1}_{[0,t]}(r)$ belongs to $\mathcal{H}_d$ and this integral exists

- Conditionally to W, $\Phi(t,x)$ is Gaussian with zero mean and

$$\mathbb{E}^B(\Phi(t,x)^2) = \alpha_{\mathbf{H}} \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d \left| W_r^i - W_s^i \right|^{2H_i-2} drds$$

- This implies

$$\mathbb{E}(\Phi(t,x)^2) = \alpha_{\mathbf{H}} \prod_{i=1}^d \mathbb{E}|\xi|^{2H_i-2} \int_0^t \int_0^t |r-s|^{2H_0+\sum_{i=1}^d H_i - d - 2} drds,$$

  where ξ is $N(0,1)$
- Therefore, $\mathbb{E}(\Phi(t,x)^2) < \infty$ if and only if $2H_0 + \sum_{i=1}^d H_i > d+1$

STEP 2

For any $\lambda \in \mathbb{R}$, we have

$$\mathbb{E}\exp\left(\lambda \int_0^t \int_{\mathbb{R}^d} \delta_0(W^x_{t-r} - y) dB_{r,y}\right) < \infty$$

- Integrating with respect to B and using the scaling properties of W it suffices to show that $\mathbb{E}(e^{\lambda Y}) < \infty$, where

$$Y = \int_0^1 \int_0^1 |s-r|^{2H_0 - 2} \prod_{i=1}^d |W^i_s - W^i_r|^{2H_i - 2} drds$$

- This follows applying Le Gall's method to derive the exponential integrability of the renormalized self-intersection local time of the planar Brownian motion

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STEP 3

The random field $u(t,x)$ satisfies Equation (1) in the weak sense: for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $t \geq 0$

$$
\begin{aligned}
\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} u_0(x)\varphi(x)dx \\
&+\frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)dxds \\
&+\int_0^t \int_{\mathbb{R}^d} u(s,x)\varphi(x) \circ dB_{s,x},
\end{aligned}
$$

where the stochastic integral is a Stratonovich integral

- The proof is based on Malliavin calculus

Remarks

(i) In the case $d = 1$, $H_1 = \frac{1}{2}$ and $H_0 > \frac{3}{4}$, for $\varphi \in \mathcal{H}_d$

$$E\left(\int_{\mathbb{R}_+ \times \mathbb{R}} \varphi_{t,x} dB_{t,x}\right)^2 = \alpha_{H_0} \int_{\mathbb{R}_+^2 \times \mathbb{R}} \varphi_{t,x} \varphi_{s,x} |s-t|^{2H_0-2} dxdsdt,$$

and the previous results can be extended

(ii) *Hölder continuity of the solution*: Suppose that $\kappa = 2H_0 + \sum_{i=1}^d H_i - d - 1 > 0$, and assume $u_0 = 1$. Then for any $\rho \in (0, \frac{\kappa}{2})$ and s, t, x, y in a compact set,

$$|u(t,y) - u(s,x)| \leq C(|t-s|^\rho + |y-x|^{2\rho})$$

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Equation in the Itô sense

Suppose that $H_0 = \frac{1}{2}$

- Feynman-Kac formula does not hold because we would need $\sum_{i=1}^{d} H_i > d$
- However, one can formulate and solve the equation in the Itô sense:

Definition

An adapted random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}\}$ is a mild solution to Equation (1) in the Itô sense if for any (t,x)

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y) \delta B_{s,y},$$

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Results

- There is a unique mild solution if $\sum_{i=1}^{d} H_i > d-1$ (and $H_i \geq \frac{1}{2}$ for $1 \leq i \leq d$). This is a particular case of a stochastic heat equation with driven by a Gaussian noise with homogeneous spacial covariance (Dalang's approach)
- The case $d = 1$ and $H_1 = \frac{1}{2}$ (*space-time white noise*) corresponds to the classical Walsh equation (continuous Anderson model):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 B}{\partial t \partial x}$$

- For $d = 2$ we need $H_1 + H_2 > 1$, so we cannot consider a space-time white noise

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Conclusions

- Itô formulas in law hold in the critical cases for different types of symmetric Riemann sums. Only the midpoint and trapezoidal Riemann sums have been considered
- Feynman-Kac formula provides a solution in the Stratonovich sense to the stochastic heat equation with a random potential which is a fractional Brownian sheet, assuming $2H_0 + \sum_{i=1}^{d} H_i > d + 1$. Open problems for this equation are:
  - Uniqueness of a weak solution
  - Asymptotic behavior as $t \to \infty$ of $\mathbb{E}(u(t,x)^p)$, where p is an integer