On Booth lemniscate and Hadamard product of analytic functions

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Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\Delta = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}\phi$. A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. In this work look at some problems on convolution and at some of the relations between the convolution and the subordination. One can consider the following problems:

Problem 1. Find the subclasses $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of the class $\mathcal{H}$ such that

$$\mathcal{A} * \mathcal{B} = \mathcal{C},$$

where $f * g$ denotes the Hadamard product or convolution of the functions f, g and

$$\mathcal{A} * \mathcal{B} = \{h(z) = (f * g)(z) \ \text{ for all } f \in \mathcal{A} \ \text{ and for all } f \in \mathcal{B}\}.$$

Problem 2. Let the sets $M \subset \mathbb{C}$ and $N \subset \mathbb{C}$ be given. Find a set $K \subset \mathbb{C}$, possible small, such that

$$f(\Delta) \subset M \ \text{ and } \ g(\Delta) \subset N \Rightarrow (f * g)(\Delta) \subset K \quad \text{for all} \quad f, g \in \mathcal{D},$$

where $\mathcal{D}$ is some subclass of $\mathcal{H}$.

In 1973 Rusheweyh and Sheil–Small proved the Pòlya–Schoenberg conjecture that the class of convex univalent functions is preserved under convolution, namely $\mathcal{K} * \mathcal{K} = \mathcal{K}$. They proved also that the class of starlike functions and the class of close-to-convex functions are closed under convolution with the class $\mathcal{K}$. An another solution of the Problem 1 is athe result due to Z. Satnkiewicz and J. Stankiewicz that

$$\mathcal{P}_\alpha * \mathcal{P}_\beta = \mathcal{P}_\gamma,$$

were $\alpha \le 1$, $\beta \le 1$, $\gamma = 1 - 2(1-\alpha)(1-\beta)$ and where

$$\mathcal{P}_x = \{p \in \mathcal{H} : p(0) = 1 \ \text{ and } \ \Re e[p(z)] > x\}.$$

The Problem 2, when the sets M and K are discs $\mathfrak{K}(s_i, R_i)$ with a center s_i and a radius R_i, was considered by K. Piejko who obtained the following theorem.

Theorem. *Let $s_i \in \mathbb{C}$, $R_i \in \mathbb{R}$; $i = 1, 2$. If $f, g \in \mathcal{H}$ and $f(0) = a$, $g(0) = b$ then the following implication holds*

$$f(\Delta) \subset \mathfrak{K}(s_1, R_1) \ \text{ and } \ g(\Delta) \subset \mathfrak{K}(s_2, R_2) \Rightarrow (f * g)(\Delta) \subset \mathfrak{K}(s, R) \quad \text{for all} \quad f, g \in \mathcal{H}$$

were

$$s = ab + m(s_1 - a)(s_2 - b), \quad R = mR_1R_2, \quad m = \frac{\left(R_1^2 - |s_1 - a|^2\right)\left(R_2^2 - |s_2 - a|^2\right)}{R_1^2R_2^2 - |s_1 - a|^2|s_2 - a|^2}.$$

In this paper we shall look for a solution of the following modified Problem 3.

Problem 4. Find a function $F \in \mathcal{H}$ and a class $\mathcal{D} \subset \mathcal{H}$ such that

$$g \prec f \quad \Rightarrow g * F \prec f * F \quad \text{for all} \quad f \in \mathcal{D}, g \in \mathcal{H},$$

where $\prec$ denotes the subordination.