ON THE COMPLETE INTEGRABILITY OF THE OSTROVSKY-VAKHNENKO EQUATION

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Abstract. The complete integrability of the Ostrovsky-Vakhnenko equation is studied by means of symplectic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, Lax type representation and related infinite hierarchy of conservation laws are constructed.

1. Introduction

The nonlinear integro-differential Ostrovsky -Vakhnenko equation

$$u_t = -uu_x - D_x^{-1}u \tag{1.1}$$

on the real axis $\mathbb{R}$ for a smooth function $u \in C^{(\infty)}(\mathbb{R}^2;\mathbb{R})$, where D_x^{-1} is the inverse-differential operator to $D_x := \partial/\partial x$, can be derived [2] as a special case of the Whitham type equation

$$u_t = -uu_x + \int_{\mathbb{R}} K(x,y)u_y dy. \tag{1.2}$$

Here the generalized kernel $K(x,y) := \frac{1}{2}|x-y|, x, y \in \mathbb{R}$ and $t \in \mathbb{R}$ is an evolution parameter. Different analytical properties of equation (1.1) were analyzed in articles [1, 2, 3], the corresponding Lax type integrability was stated in [5].

Recently by J.C. Brunelli and S. Sakovich in [4] there was demonstrated that Ostrovsky - Vakhnenko equation is a suitable reduction of the well known Camassa-Holm equation that made it possible to construct the corresponding compatible Poisson structures for (1.1), but in a complicated enough non-polynomial form.

In the present work we will reanalyze the integrability of equation (1.1) both from the gradient-holonomic [7, 11, 12], symplectic and formal differential-algebraic points of view. As a result, we will rederive the Lax type representation for the Ostrovsky -Vakhnenko equation (1.1), construct the related simple enough compatible polynomial Poisson structures and an infinite hierarchy of conservation laws.

2. Gradient-holonomic integrability analysis

Consider the nonlinear Ostrovsky -Vakhnenko equation (1.1) as a suitable nonlinear dynamical system

$$du/dt = -uu_x - D_x^{-1}u := K[u] \tag{2.1}$$

on the smooth 2π-periodic functional manifold

$$M := \{u \in C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}) : \int_0^{2\pi} udx = 0\}, \tag{2.2}$$

where $K : M \to T(M)$ is the corresponding well-defined smooth vector field on M.

We, first, will state that the dynamical system (2.1) on manifold M possesses an infinite hierarchy of conservation laws, that can signify as a necessary condition for its integrability. For this we need to construct a solution to the Lax gradient equation

$$\varphi_t + K^{',*}\varphi = 0, \tag{2.3}$$

in the special asymptotic form

$$\varphi = \exp[-\lambda t + D_x^{-1}\sigma(x;\lambda)], \tag{2.4}$$

where, by definition, a linear operator $K^{',*} : T^*(M) \to T^*(M)$ is, adjoint with respect to the standard convolution $(\cdot,\cdot)$ on $T^*(M) \times T(M)$, the Frechet-derivative of a nonlinear mapping

$K : M \to T(M)$:

$$K^{',*} = uD_x + D_x^{-1} \tag{2.5}$$

and, respectively,

$$\sigma(x;\lambda) \simeq \sum_{j\in\mathbb{Z}_+} \sigma_j[u]\lambda^{-j}, \tag{2.6}$$

as $|\lambda| \to \infty$ with some "local" functionals $\sigma_j : M \to C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ on M for all $j \in \mathbb{Z}_+$.

By substituting (2.4) into (2.3) one easily obtains the following recurrent sequence of functional relationships:

$$\sigma_{j,t} + \sum_{k\le j} \sigma_{j-k}(u\sigma_k + D_x^{-1}\sigma_{k,t}) - \sigma_{j+1} + (u\sigma_j)_x + \delta_{j,0} = 0 \tag{2.7}$$

for all $j+1 \in \mathbb{Z}_+$ modulo the equation (2.1). By means of standard calculations one obtains that this recurrent sequence is solvable and

$$\begin{aligned} \sigma_0[u] &= 0, \sigma_1[u] = 1, \sigma_2[u] = u_x, \\ \sigma_3[u] &= 0, \sigma_4[u] = u_t + 2uu_x, \\ \sigma_5[u] &= 3/2(u^2)_{xt} + u_{tt} + 2/3(u^3)_{xx} - u_x D_x^{-1}u \end{aligned} \tag{2.8}$$

and so on. It is easy check that all of functionals

$$\gamma_j := \int_0^{2\pi} \sigma_j[u]dx \tag{2.9}$$

are on the manifold M conservation laws, that is $d\gamma_j/dt = 0$ for $j \in \mathbb{Z}_+$ with respect to the dynamical system (2.1). For instance, at $j = 5$ one obtains:

$$\begin{aligned} \gamma_5 \quad : &= \int_0^{2\pi} \sigma_5[u]dx = \int_0^{2\pi} \left[3/2(u^2)_{xt} + u_{tt} + 2/3(u^3)_{xx} - u_x D_x^{-1}u\right] dx = \\ &= \frac{d^2}{dt^2}\int_0^{2\pi} u_{tt}dx - \int_0^{2\pi} u_x D_x^{-1}udx = \frac{d^2}{dt^2}\int_0^{2\pi} udx - uD_x^{-1}u\Big|_0^{2\pi} + \int_0^{2\pi} u^2dx = \\ &= \int_0^{2\pi} u^2dx, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} d\gamma_5/dt &= 2\int_0^{2\pi} uu_tdx = -2\int_0^{2\pi} u(uu_x + D_x^{-1}u)dx = \\ &= -2\int_0^{2\pi} uD_x^{-1}udx = -\int_0^{2\pi} [(D_x^{-1}u)^2]_x dx = (D_x^{-1}u)^2\Big|_0^{2\pi} = 0, \end{aligned} \tag{2.11}$$

since owing to the constraint (2.2) the integrals $(D_x^{-1}u)\Big|_0^{2\pi} = 0$.

The result stated above allows us to suggest that the dynamical system (2.1) on the functional manifold M is an integrable Hamiltonian system.

First, we will show that this dynamical system is a Hamiltonian flow

$$du/dt = -\vartheta grad\ H[u] \tag{2.12}$$

with respect to some Poisson structure $\vartheta : T^*(M) \to T(M)$ and a Hamiltonian function $H \in \mathcal{D}(M)$. Based on the standard symplectic techniques [10, 7, 6, 11] consider the conservation law (2.10) and present it in the scalar "momentum" form:

$$-1/2\gamma_5 = \frac{1}{2}\int_0^{2\pi} u_x D_x^{-1}udx = (1/2D_x^{-1}u, u_x) := (\psi, u_x) \tag{2.13}$$

with the co-vector $\psi := 1/2D_x^{-1}u \in T^*(M)$ and calculate the corresponding co-Poissonian structure

$$\vartheta^{-1} := \psi' - \psi'^{,*} = D_x^{-1}, \tag{2.14}$$

or the Poissonian structure

$$\vartheta = D_x. \tag{2.15}$$

The obtained operator $\vartheta = D_x : T^*(M) \to T(M)$ is really Poissonian for (2.1) since the following determining symplectic condition

$$\psi_t + K^{',*}\psi = grad\ \mathcal{L} \tag{2.16}$$

holds for the Lagrangian function

$$\mathcal{L} = \frac{1}{4}\int_0^{2\pi}[u^3/3 + (D_x^{-1}u)^2]dx. \tag{2.17}$$

As a result of (2.16) one obtains easily that

$$du/dt = -\vartheta grad\ H[u], \tag{2.18}$$

where the Hamiltonian function

$$H = (\psi, K) - \mathcal{L} = \frac{1}{2}\int_0^{2\pi}[u^3/3 - (D_x^{-1}u)^2/2]dx \tag{2.19}$$

is an additional conservation law of the dynamical system (2.1). Thus, one can formulate the following proposition.

Proposition 2.1. *The Ostrovsky-Vakhnenko dynamical system (2.1) possesses an infinite hierarchy of nonlocal, in general, conservation laws (2.9) and is a Hamiltonian flow (2.18) on the manifold M with respect to the Poissonian structure (2.15).*

Remark 2.2. It is useful to remark here that the existence of an infinite ordered by λ-powers hierarchy of conservations laws (2.9) is a typical property [10, 6, 7, 11] of the Lax type integrable Hamiltonian systems, which are simultaneously bi-Hamiltonian flows with respect to corresponding two compatible Poissonian structures.

As is well known [10, 6, 7, 11], the second Poissonian structure $\eta : T^*(M) \to T(M)$ on the manifold M for (2.1), if it exists, can be calculated as

$$\eta^{-1} := \tilde{\psi}' - \tilde{\psi}'^{,*}, \tag{2.20}$$

where a-covector $\tilde{\psi} \in T^*(M)$ is a second solution to the determining equation (2.16):

$$\tilde{\psi}_t + K^{',*}\tilde{\psi} = grad\ \tilde{\mathcal{L}} \tag{2.21}$$

for some Lagrangian functional $\tilde{\mathcal{L}} \in \mathcal{D}(M)$. It can be certainly done by means of simple enough but cumbersome analytical calculations based, for example, on the asymptotical small parameter method [7, 11, 12] and on which we will not stop here.

Instead of this we will shall apply the direct differential-algebraic approach to dynamical system (2.1) and reveal its Lax type representation both in the differential scalar and in canonical matrix Zakharov-Shabat forms. Moreover, we will construct the naturally related compatible polynomial Poissonian structures for Ostrovsky -Vakhnenko dynamical system (2.1) and generate an infinite hierarchy of commuting to each other nonlocal conservation laws.

3. Lax type representation and compatible Poissonian structures

We will start with construction of the polynomial differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x,t\}\}$ generated by a fixed functional variable $u \in \mathbb{R}\{\{x,t\}\}$ and invariant with respect to two differentiations $D_x := \partial/\partial x$ and $D_t := \partial/\partial t + u\partial/\partial x$, satisfying the Lie-algebraic commutator relationship

$$[D_x, D_t] = u_x D_x. \tag{3.1}$$

Since the Lax type representation for the dynamical system (2.1) can be interpreted [8, 11] as the existence of a finite-dimensional invariant ideal $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$, realizing the corresponding finite-dimensional representation of the the Lie-algebraic commutator relationship (3.1), this ideal can be presented as

$$\mathcal{I}\{u\} := \{\sum_{j\in\mathbb{Z}_+} g_j D_x^j f[u] \in \mathcal{K}\{u\} : g_j \in \mathcal{K}, j \in \mathbb{Z}_+\}, \tag{3.2}$$

where an element $f[u] \in \mathcal{K}\{u\}$ is fixed. The D_x-invariance of ideal (3.2) is *a priori* evident, but its D_t-invariance strongly depends on the element $f[u] \in \mathcal{K}\{u\}$, which can be found from the following functional relationship on the element $\varphi[u;\lambda] := grad\ \gamma(\lambda) \in \mathcal{K}\{u\}, \gamma(\lambda) := \int_0^{2\pi} \sigma(x;\lambda)dx$:

$$D_t\varphi = -D_x^{-1}\varphi. \tag{3.3}$$

Really, let us define the following element

$$\overline{\varphi} := D_x^{-2}\varphi \tag{3.4}$$

and represent it in a factorized form as

$$\overline{\varphi} := \tilde{f}f, \tag{3.5}$$

where functions $\tilde{f}, f \in \mathcal{K}\{u\}$ satisfy the adjoint pair of differential relationships:

$$\begin{aligned} D_t f &= a_j f + a_1 D_x f + a_2 D_x^2 f\ , \\ D_t \tilde{f} &= -a_0 f + D_x(a_1 f) - D_x^2(a_2 f), \end{aligned} \tag{3.6}$$

for some elements $a_j \in \mathcal{K}\{u\}, j = \overline{0,2}$.

The second order of differential relationships (3.6) is motivated by the following expression

$$\varphi = D_x^2\bar{\varphi} = D_x^2(\tilde{f}f), \tag{3.7}$$

where the differentials elements $f, D_x f, D_x^2 f$ and $\tilde{f}, D_x\tilde{f}, D_x^2\tilde{f}$ are considered, respectively, functionally independent in the ideal $\mathcal{K}\{u\}$.

This, in particular, means that there exists the corresponding differential relationships

$$\begin{aligned} D_x^3 f &= b_0 f + b_1 D_x f + b_2 D_x^2 f\ , \\ D_x^3 \tilde{f} &= -b_0 \tilde{f} + D_x(b_1 \tilde{f}) - D_x^2(b_2 f), \end{aligned} \tag{3.8}$$

for some elements $b_j \in \mathcal{K}\{u\}, j = \overline{0,2}$, allowing to reduce the ideal (3.2) to the form

$$\mathcal{I}_3\{u\} := \{\sum_{j=0}^{2} g_j D_x^j f[u] \in \mathcal{K}\{u\} : g_j \in \mathcal{K}, j \in \overline{0,2}\}, \tag{3.9}$$

which is, owing to (3.6) and (3.8), D_x- and D_t - invariant, if the compatibility condition (3.3) holds. The latter is checked by means of simple enough calculations giving rise to the following result concerning the expressions (3.6):

$$D_t f = \mu^{-1} D_x^2 f + u_x f,\ D_t\tilde{f} = \mu^{-1} D_x^2 \tilde{f} - u_x \tilde{f}, \tag{3.10}$$

and the expressions (3.8):

$$D_x^3 f = -\mu\bar{u}f,\ D_x^3\tilde{f} = \mu\bar{u}\tilde{f}, \tag{3.11}$$

where $\bar{u} := u_{xx} + 1/3, \mu \in \mathbb{C}\backslash\{0\}$ is an arbitrary complex parameter, and coinciding with those found before in [5]. Moreover, the compatibility condition (3.3) gives rise to the important relationship

$$D_x^2 D_t\varphi - \vartheta\varphi = 3\mu^2\eta\varphi, \tag{3.12}$$

where the polynomial integro-differential operator

$$\eta := \partial^{-1}\bar{u}\partial^{-3}\bar{u}\partial^{-1} + 4\partial^{-2}\bar{u}\partial^{-1}\bar{u}\partial^{-2} + 2(\partial^{-2}\bar{u}\partial^{-2}\bar{u}\partial^{-1} + \partial^{-1}\bar{u}\partial^{-2}\bar{u}\partial^{-2}) \tag{3.13}$$

is skewsymmetric on M and presents the second compatible Poisson structure for the Ostrovsky-Vachnenko dynamical system (2.1).

Remark 3.1. It is interesting to observe that our second polynomial Poisson structure (3.13) differs from that obtained recently in [4], which contains the irrational power factors.

Based now on the recurrent relationships following from substitution of the asymptotic expansion

$$\varphi :\simeq \sum_{j\in\mathbb{Z}_+} \varphi_j\lambda^{-j},\ \lambda := -1/(3\mu^2), \tag{3.14}$$

in to (3.11), one can determine a new infinite hierarchy of conservations laws for dynamical system (2.1):

$$\tilde{\gamma}_j := \int_0^1 d\lambda(\varphi_j[u\lambda], u), \tag{3.15}$$

for $j \in \mathbb{Z}_+$, where

$$\varphi_j = \Lambda^j \varphi_0, \ \nu\varphi_0 = 0, \tag{3.16}$$

and the recursion operator $\Lambda := \vartheta^{-1}\eta : T^*(M) \to T^*(M)$ satisfies the standard Lax type representation:

$$\Lambda_t = [\Lambda, K^{'*}]. \tag{3.17}$$

The obtained above results can be formulated as follows.

Proposition 3.2. *Proposition 3.1. The Ostrovsky-Vachnenko dynamical system (2.1) allows the standard Lax type representation (3.10), (3.11) and defines on the functional manifold M an integrable bi-Hamiltonian flow with compatible Poisson structures (2.15) and (3.13). In particular, this dynamical system possesses an infinite hierarchy of nonlocal conservation laws (3.15), defined by the gradient elements (3.16).*

It is easy to construct making use of the differential expressions (3.10) and (3.11) the standard matrix Zakharov-Shabat form of the Lax type representation for the dynamical system (1.1).

Really, if to define the "spectral" parameter $\mu := 1/(9\lambda) \in \mathbb{C}\backslash\{0\}$ and new basis elements of the ideal $\mathcal{I}_3\{u\}$:

$$g_1 := -3D_x f, \ g_2 := f, \ g_3 := 9\lambda D_x^2 f + u_x f, \tag{3.18}$$

then relationships (3.10)and (3.11) can be rewritten as follows:

$$D_t g = q[u;\lambda]g, \ D_x g = l[u;\lambda]g, \tag{3.19}$$

where matrices

$$q[u;\lambda] := \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -u & 0 \end{array}\right), l[u;\lambda] := \left(\begin{array}{ccc} 0 & u_x/(3\lambda) & -1/(3\lambda) \\ -1/3 & 0 & 0 \\ -u_x/3 & -1/3 & 0 \end{array}\right) \tag{3.20}$$

coincide with those of [5, 4] and satisfy the following Zakharov-Shabat type compatibility condition:

$$D_t l = [q,l] + D_x q - l \ D_x u.$$

4. A related Riemann type hydrodynamic system and its differential-algebraic analysis

It is now worth to observe that the Ostrovsky-Vakhnenko dynamical system (2.1) allows the following equivalent recurrent Riemann type hydrodynamic representation:

$$D_t^2 u = z, \qquad z_0 + u^2 := ku, \tag{4.1}$$

for some $k \in \mathbb{R}$, where the differentiations $D_t := \partial/\partial t + u\partial/\partial x$ and $D_x : \mathbb{R}\{\{x,t\}\} \to \mathbb{R}\{\{x,t\}\}$ of the functional ring $\mathbb{R}\{\{x,t\}\}$ satisfy the previous Lie-algebraic commutator relationship (3.1). The system (4.1) can also be rewritten in a more convenient form as

$$D_t u = v, D_t v = z, \quad z + u^2 := ku. \tag{4.2}$$

The latter form is well fitting for studying its integrability by means of the differential-algebraic tools, devised before in [8, 9].

Namely, it is natural to construct in the ring $\mathcal{K}\{[u]\}$,where $[u] := \{u_1, u_2, ..., u_n, ...\} \subset \mathbb{R}\{\{x,t\}\}$ is an arbitrary but fixed set of functions, the following *a priori* D_x- and D_t-invariant parametrically dependent on $\lambda \in \mathbb{R}$ differential ideal:

(4.3)
$$\mathcal{I}\{[u]\} := \{\sum_{n\in\mathbb{Z}_+}\sum_{m=\overline{1,2}} f_n^{(m)}\lambda^{-m}D_t^m u_n \in \mathcal{K}\{[u]\} : f_n^{(m)} \in \mathbb{R}\{\{x,t\}\}, m = \overline{1,3}, n \in \mathbb{Z}_+, \lambda \in \mathbb{R}\},$$

and study the kernel $\ker D_t \subset \mathcal{I}\{[u]$ of the corresponding differentiation $D_t : \mathcal{I}\{[u]\}\} \to \mathcal{I}\{[u]\}$. As a result of simple calculations one obtains that the kernel $\ker D_t \subset \mathcal{I}\{[u]$ is generated by an infinite-dimensional vector $f \in \mathbb{R}^\infty\{\{x,t\}\}$ satisfying the following differential relationship:

$$\begin{array}{rcl} D_t f & = & q(\lambda) f, \\ q(\lambda) & : & = \left(\begin{array}{c} \\ \\ \\ \end{array}\right) \end{array}$$

5. Conclusion

We have showed that the Ostrovsky-Vachnenko dynamical system is naturally embedded into the general Lax type integrability scheme [10, 6, 7, 11] whose main ingredient such as the corresponding compatible Poissonian structures and Lax type representation can be retrieved by means of different direct modern integrability tools, such as the differential-geometric, differential-algebraic and symplectic gradient holonomic approaches.

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