Grothendieck-Lidskiĭ theorem for subspaces and factor spaces of L_p-spaces

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Abstract. In 1955, A. Grothendieck has shown that if the linear operator T in a Banach subspace of an L_∞-space is 2/3-nuclear then the trace of T is well defined and is equal to the sum of all eigenvalues $\{\mu_k(T)\}$ of T. V.B. Lidskiĭ , in 1959, proved his famous theorem on the coincidence of the trace of the S_1-operator in $L_2(\nu)$ with its spectral trace $\sum_{k=1}^{\infty} \mu_k(T)$. We show that *for $p \in [1, \infty]$ and $s \in (0, 1]$ with $1/s = 1 + |1/2 - 1/p|$, and for every s-nuclear operator T in every subspace of any $L_p(\nu)$-space the trace of T is well defined and equals the sum of all eigenvalues of T.* Note that for $p = 2$ one has $s = 1$, and for $p = \infty$ one has $s = 2/3$.

In 1955, A. Grothendieck [1] has shown that if the linear operator T in a Banach space is 2/3-nuclear then the trace of T is well defined and is equal to the sum of all eigenvalues $\{\mu_k(T)\}$ of T. V.B. Lidskiĭ [2], in 1959, proved his famous theorem on the coincidence of the trace of the S_1-operator in an (infinite dimensional) Hilbert space with its spectral trace $\sum_{k=1}^{\infty} \mu_k(T)$. Any Banach space is a subspace of an $L_\infty(\nu)$-space, as well as any Hilbert space is a (subspace of) $L_2(\nu)$-space. Also, any Banach space is a factor space of an $L_1(\nu)$-space, as well as any Hilbert space is a (factor space of) $L_2(\nu)$-space. We obtain the following generalization of these theorems: *for $p \in [1, \infty]$ and $s \in (0, 1]$ with $1/s = 1 + |1/2 - 1/p|$, and for every s-nuclear operator T in every subspace of any $L_p(\nu)$-space the trace of T is well defined and equals the sum of all eigenvalues of T.* Note that for $p = 2$ one has $s = 1$, and for $p = \infty$ one has $s = 2/3$.

§1. Definitions and a theorem

All the terminology and facts (now classical), given here without any explanations, can be found in [7–10].

Let X, Y be Banach spaces. For $s \in (0, 1]$, denote by $X^*\widehat{\otimes}_s Y$ the completion of the tensor product $X^* \otimes Y$ (considered as a linear space of all finite rank operators)

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with respect to the quasi-norm

$$||z||_s := \inf\{ \left(\sum_{k=1}^{N} ||x'_k||^s \, ||y_k||^s \right)^{1/s} \; : \; z = \sum_{k=1}^{N} x'_k \otimes y_k \}.$$

Let Φ_p, for $p \in [1, \infty]$, be the ideal of all operators which can be factored through a subspace af an L_p-space. Put $N_s(X, Y) :=$ image of $X^* \widehat{\otimes}_s Y$ in the space $L(X, Y)$ of all bounded linear transformations under the canonical factor map $X^* \widehat{\otimes}_s Y \to N_s(X, Y) \subset L(X, Y)$. We consider the (Grothendieck) space $N_s(X, Y)$ of all s-nuclear operators from X to Y with the natural quasi-norm, induced from $X^* \widehat{\otimes}_s Y$.

Finally, let $\Phi_{p,s}$ (respectively, $\Phi_{s,p}$) be the quasi-normed product $N_s \circ \Phi_p$ (respectively, $\Phi_p \circ N_s$) of the corresponding ideals equipped with the natural quasi-norm $\nu_{p,s}$ (respectively, $\nu_{s,p}$): if $A \in N_s \circ \Phi_p(X, Y)$ then $A = \varphi \circ T$ with $T = \beta\alpha \in \Phi_p$, $\varphi = \delta\Delta\gamma \in N_s$ and

$$A : X \xrightarrow{\alpha} X_p \xrightarrow{\beta} Z \xrightarrow{\gamma} c_0 \xrightarrow{\Delta} l_1 \xrightarrow{\delta} Y,$$

where all maps are continuous and linear, X_p is a subspace of an L_p-space, constructed on a measure space, and Δ is a diagonal operator with the diagonal from l_s. Thus, $A = \delta\Delta\gamma\beta\alpha$ and $A \in N_s$. Therefore, if $X = Y$, the spectrum of A, $sp\,(A)$, is at most countable with only possible limit point zero. Moreover, A is a Riesz operator with eigenvalues of finite algebraic multiplicities and $sp\,(A) \equiv sp\,(B)$, where $B := \alpha\delta\Delta\gamma\beta : X_p \to X_p$ is an s-nuclear operator, acting in a subspace of an L_p-space.

Let T be an operator between Banach spaces Y and W. The operator $\mathbf{1} \otimes T : X^* \widehat{\otimes}_s Y \to X^* \widehat{\otimes}_s W$ is well defined and can be considered also as an operator from $X^* \widehat{\otimes}_s Y$ into $X^* \widehat{\otimes} W$ (the Grothendieck projective tensor product), the last space having the space $L(W, X^{**})$ as dual.

Definition. We say that T *possesses the property* AP_s (written down as "$T \in AP_s$") if for every X and any tensor element $z \in X^* \widehat{\otimes}_s Y$ the operator $T \circ z : X \to W$ is zero iff the corresponding tensor $(\mathbf{1} \otimes T)(z)$ is zero as an element of the space $X^* \widehat{\otimes} W$. If $Y = W$ and T is the identity map, we write just $Y \in AP_s$ (the approximation property of order s).

This is equivalent to the fact that if $z \in X^* \widehat{\otimes}_s Y$ then it follows from

$$\operatorname{trace}\,(\mathbf{1} \otimes T)(z) \circ R = 0, \quad \forall\, R \in W^* \otimes X$$

that trace $U \circ (\mathbf{1} \otimes T)(z) = 0$ for every $U \in L(W, X^{**})$. There is a simple characterization of the condition $T \in AP_s$ in terms of the approximation of T on some sequences of the space Y, but we omit it now, till the next time. We need here only one example which is crucial for our note (other examples, as well as more general applications will appear elsewhere).

Example. Let $s \in (0, 1]$, $p \in [1, \infty]$ and $1/s = 1 + |1/p - 1/2|$. Any subspace as well as any factor space of any L_p-space have the property AP_s (this means that, for that space Y, $\mathrm{id}_Y \in AP_s$). Thus, in the case of such a space Y, we have the quasi-Banach equality $X^* \widehat{\otimes}_s Y = N_s(X, Y)$, whichever the space X was.

Lemma. Let $s \in (0,1]$, $p \in [1,\infty]$ and $1/s = 1 + |1/2 - 1/p|$. Then the system of all eigenvalues (with their algebraic multiplicities) of any operator $T \in N_s(Y,Y)$, acting in any subspace Y of any L_p-space, belongs to the space l_1. The same is true for the factor spaces of L_p-spaces.

Corollary. If $s \in (0,1]$, $p \in [1,\infty]$ with $1/s = 1 + |1/2 - 1/p|$ then the quasi-normed ideals $\Phi_{p,s}$ and $\Phi_{s,p}$ are of (spectral) type l_1.

Theorem. Let Y be a subspace or a factor space of an L_p-space, $1 \le p \le \infty$. If $T \in N_s(Y,Y)$, $1/s = 1 + |1/2 - 1/p|$, then

1. the (nuclear) trace of T is well defined,
2. $\sum_{n=1}^{\infty} |\lambda_n(T)| < \infty$, where $\{\lambda_n(T)\}$ is the system of all eigenvalues of the operator T (written in according to their algebraic multiplicities)

and

$$\text{trace } T = \sum_{n=1}^{\infty} \lambda_n(T).$$

§2. Proofs

Proof of Lemma. Let Y be a subspace or a factor space of an L_p-space and $T \in N_s(Y,Y)$ with an s-nuclear representation

$$T = \sum_{k=1}^{\infty} \mu_k y'_k \otimes y_k,$$

where $||y'_k||, ||y_k|| = 1$ and $\mu_k \ge 0$, $\sum_{k=1}^{\infty} \mu_k^s < \infty$. The operator T can be factored in the following way:

$$T: Y \xrightarrow{A} l_\infty \xrightarrow{\Delta_{1-s}} l_r \overset{j}{\hookrightarrow} c_0 \xrightarrow{\Delta_s} l_1 \xrightarrow{B} Y,$$

where A and B are linear bounded, j is the natural injection, $\Delta_s \sim (\mu_k^s)_k$ and $\Delta_{1-s} \sim (\mu_k^{1-s})$ are the natural diagonal operators from c_0 into l_1 and from l_∞ into l_r, respectively. Here, r is defined via the conditions $1/s = 1 + |1/p - 1/2|$ and $\sum_k \mu_k^s < \infty$: we have to have $\sum_k \mu_k^{(1-s)r} < \infty$, for which $(1-s)r = s$ is good. Therefore, put $1/r = 1/s - 1$, or $1/r = |1/p - 1/2|$.

from now onward in the proof we assume (surely, without loss of generality) that $p \ge 2$. Then $1/r = 1/2 - 1/p$ and $r(1-s) = s$. Note that if $s = 1$ then $r = \infty, p = 2$ and $j\Delta_{1-s} \equiv j$; and if $s = 2/3$ then $r = 2, p = \infty$ and $\Delta_{1-s} \sim (\mu_k^{1/3})_k \in l_2$.

Now, let us factorize the diagonal Δ_s as $\Delta_s = \Delta_2\Delta_1 : c_0 \xrightarrow{\Delta_1} l_2 \xrightarrow{\Delta_2} l_1$ in such a (clear) way that diagonals Δ_1 is in Π_2 and Δ_2^* is in Π_2 too, respectively.

Case (i). Y is a subspace of an L_p-space. Denoting by $l : Y \hookrightarrow L_p$ an isomorphic embedding of Y into a corresponding $L_p = L_p(\nu)$, we obtain that the map $\Delta_2^* B^* l^*$: $L_{p'} \xrightarrow{l^*} Y^* \xrightarrow{B^*} l_\infty \xrightarrow{\Delta_2^*} l_2$ is of type Π_2, so is in Π_p. Thus its preadjoint $lB\Delta_2$: $l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{B} Y \xrightarrow{l} L_p$ is order bounded and, therefore, p-absolutely summing.

Case (ii). Y is a factor space of an L_p-space. Denoting by $q : L_p \to Y$ a factor map from a corresponding $L_p = L_p(\nu)$ onto Y and taking a lifting $Q : l_1 \to L_p$ for B with $B = qQ$, we obtain that the map $\Delta_2^* Q^*$: $L_{p'} \xrightarrow{Q^*} l_\infty \xrightarrow{\Delta_2^*} l_2$ is of type Π_2, so is in Π_p. Thus its pre-adjoint $Q\Delta_2 : \, l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{Q} L_p$ is order bounded and, therefore, p-absolutely summing. Hence, $B\Delta_2 : \, l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{Q} L_p \xrightarrow{q} Y$ is also p-absolutely summing.

It follows from all that's said that in all the cases our operator $T : Y \to Y$ can be written as a composition:

$$T = U_1 U_2 U_3 \;\text{ with }\; U_3 \in \Pi_r, U_2 \in \Pi_2, U_1 \in \Pi_p,$$

all the exponents being not less than 2. Now, $1/r + 1/2 + 1/p = (1/2 - 1/p) + 1/2 + 1/p = 1$.

Proof of the statement of Example. It follows from:

(α) (see, e.g., [14] or [16]) every finite dimensional subspace E of any factor space of any L_p-space is $c_p\,(\dim E)^{|1/2-1/p|}$-complemented.

For more general statements on AP_s and their proofs, we refer to [4] and [5]; see also an old paper of O.I. Reinov [3] for the idea to apply the projections in the questions which are under consideration in this note.

Proof of Corollary. Apply Lemma.

Proof of Theorem. (1). Let Y be a subspace of an L_p-space and $T \in N_s(Y, Y)$. By the assertion of Example, we may (and do) identify the space $N_s(Y, Y)$ with the corresponding tensor product $Y^* \widehat{\otimes}_s Y$, which, in turn, is a subspace of the projective tensor product $Y^* \widehat{\otimes} Y$. Thus, the nuclear trace of T is well defined, and we have to show that this trace of T is just the spectral trace (= spectral sum) $\sum_{n=1}^{\infty} \lambda_n(T)$.

By Lemma, the sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$ of all eigenvalues of T, counting by multiplicities, is in l_1. Since the quasi-normed ideals $\Phi_{p,s}$ and $\Phi_{s,p}$ are of spectral (= eigenvalue) type l_1 (see Corollary), we can apply the main result from the paper [6] of M.C. White, which asserts:

$(**)$ *If J is a quasi-Banach operator ideal with eigenvalue type l_1, then the spectral sum is a trace on that ideal J.*

Recall (see [10], 6.5.1.1, or Definition 2.1 in [6]) that a *trace* on an operator ideal J is a class of complex-valued functions, all of which one writes as τ, one for each component $J(E, E)$, where E is a Banach space, so that

(i) $\tau(e' \otimes e) = \langle e', e \rangle$ for all $e' \in E^*, e \in E$;

(ii) $\tau(AU) = \tau(UA)$ for all Banach spaces F and operators $U \in J(E, F) and A \in L(F, E)$;

(iii) $\tau(S + U) = \tau(S) + \tau(U)$ for all $S, U \in J(E, E)$;

(iv) $\tau(\lambda U) = \lambda \tau(U)$ for all $\lambda \in \mathbb{C}$ and $U \in J(E, E)$.

Our operator T, evidently, belongs to the space $\Phi_{s,p}(Y, Y)$ and, as was said, $\Phi_{s,p}$ is of eigenvalue type l_1. Thus, the assertion $(**)$ implies that the spectral sum λ, defined by $\lambda(U) := \sum_{n=1}^{\infty} \lambda_n(U)$ for $U \in \Phi_{s,p}(E, E)$, is a trace on $\Phi_{s,p}$.

By principle of uniform boundedness (see [11], 3.4.6 (page 152), or [12]), there exists a constant $C > 0$ with the property that

$$|\lambda(U)| \le ||\{\lambda_n(U)\}||_{l_1} \le C\, \nu_{s,p}(U)$$

for all Banach spaces E and operators $U \in \Phi_{s,p}(E, E)$.

Now, remembering that all operators in $\Phi_{s,p}$ can be approximated by finite rank operators and taking in account the conditions (iii)–(iv) for $\tau = \lambda$, we obtain that the nuclear trace of our operator T coincides with $\lambda(T)$ (recall that the continuous trace is uniquely defined in such a situation; see [10], 6.5.1.2).

(2). If Y is a quotient of an L_p-space, we can apply the fact that, for a Riesz operator $U \in L(E, E)$, the adjoint U^* is also a Riesz operator and these two operators have the same eigenvalues $\lambda \neq 0$ with the same multiplicities (see, e.g., [11], Theorem 3.2.26, or [13], Exercise VII.5.35). So, this part (2) follows immediately from the just proved part (1). If one wishes, we can use, in the cases where $p = 1$ or $p = \infty$, also the fact that *every* Banach space is a quotient of an L_1-space and apply either the Grothendieck's 2/3-theorem [1] or its reproved version from [3].

Remark 1: Since finite rank operators are dense in N_s, Theorem can be proved without referring to the paper of M.C. White; but this would take a little bit longer explanations.

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