Approximate (Abelian) groups

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4th July 2012

The 99% question

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$A \subset G$ is a coset (of a subgroup) in G iff $A \neq \emptyset$ and

$$x, y, z \in A \Rightarrow x + y - z \in A.$$

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Rough question

What if only 99% of triples $x, y, z \in A$ have $x + y - z \in A$?

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$$E(A) := \#\{(x, y, z) \in A^3 : x + y - z \in A\}.$$

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- Easy: $E(A) \leq |A|^3$.
- Before: $A \subset G$ is a coset in G iff $E(A) = |A|^3$.
- Think of A finite, $|A| \to \infty$ and $\epsilon \lesssim 1/100$.

Example

H is a coset in G and A has

- $|A \cap H| \geq (1-\eta)|A|$;
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Short calculation:

$$E(A) \geq (1 - O(\eta))|A|^3.$$

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Proposition

Suppose that $E(A) \geq (1-\epsilon)|A|^3$. Then there is some coset H in G such that

- $|A \cap H| \geq (1 - O(\epsilon^{1/2}))|A|$;
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- Weakness: these approximate groups are all close to actual groups.

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Idea for examples: Q a convex body in $\mathbb{R}^d$ *e.g.* a cube. Then

$$\mathbb{P}(x + y - z \in Q | x, y, z \in Q) \geq \exp(-O(d)).$$

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In some sense '$E(Q) \geq \exp(-O(d))|Q|^3$'.

Examples for the 1% question

Convex progressions

A *d-dimensional convex progression* in G is a set of the form $\phi(Q \cap \mathbb{Z}^d)$ where

- Q is a symmetric convex body in $\mathbb{R}^d$;
- and $\phi : \mathbb{Z}^d \to G$ is a homomorphism.

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Convex coset progressions

A *d-dimensional convex coset progression* in G is then a set $H + P$ where

- P is a d-dimensional convex progression;
- and H is a coset in G.

Examples for the 1% question

Example

M is a convex coset progression and A is any set such that

- $|A \cap M| \geq \exp(-d)|A|$;
- $|A \cap M| \geq \exp(-d)|M|$;
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Suppose that $E(A) \geq \delta|A|^3$. Then there is a convex coset progression M such that

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- G arbitrary (Abelian) Green and Ruzsa.

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- Output useful: convex coset progressions support (rough) harmonic analysis.
- Rough equivalence: any set satisfying the conclusion also satisfies the hypothesis with δ replaced by $\exp(-O(d(\delta)))$.

Quality of the rough equivalence in Freĭman's theorem

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- Polynomial-Freĭman-Ruzsa conjecture: $d(\delta) = O(\log \delta^{-1})$.

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- (Independent copies of *different* subgroups) $k \sim \delta^{-1/2}$ and $H_1, \ldots, H_k$ are 'totally different' subgroups of same size. $A = \bigcup_{i=1}^{k} H_i$ has $E(A) \approx \delta|A|^3$.

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Argument de-couples into three parts.

De-coupling the argument: step 1

Balog-Szemerédi-Gowers lemma

Suppose $E(A) \geq \delta|A|^3$. Then there is $A' \subset A$ with

$$|A'| \geq \delta^{O(1)}|A| \text{ and } |A' - A'| \leq \delta^{-O(1)}|A'|.$$

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- Eliminates the third structure-type.
- Polynomial bounds.
- Proof (Gowers) by dependent random choice.

De-coupling the argument: step 2

Easy: d-dimensional convex coset progression M has relative polynomial growth meaning

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Convex coset progressions and relative polynomial growth

Suppose $|nX| \leq n^d|X|$ for all $n \geq 1$. Then X 'is' a $d^{1+o(1)}$-dimensional convex coset progression.

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- Proof via harmonic analysis and geometry of numbers.

Decoupling the argument: step 3

Aim

Show that if $|A-A| \leq K|A|$ then there is some X such that

- $|A \cap X| \geq \exp(-d'(K))|A|$;
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- Leads to Freĭman's theorem with $d(\delta) = d'(\delta^{-O(1)})$.

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Argument has two main ideas.

López-Ross

Write $f * \mu_A(x)$ for the average value of f on the set $x - A$.

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López and Ross

If $x \in A$ then average value of 1_{A-A} on $x - A$ is 1, so

$$\langle 1_{A-A} * \mu_A, 1_A \rangle = 1.$$

Croot-Sisask

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If $|A - A| \leq K|A|$ then there is X large such that

$$\left(\sum_{y \in G} |f * \mu_A(y + x) - f * \mu_A(y)|^p\right)^{1/p} \leq \epsilon \left(\sum_{y \in G} |f(y)|^p\right)^{1/p}$$

for all $x \in X$.

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for all $x \in X$.

In words: $f * \mu_A$ does not vary much in ℓ^p when translating by elements of X.

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The first condition leads to

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The second gives the relative polynomial growth.