Coulomb gases, Abrikosov lattice, and renormalized energy

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Three models

1. energy of a 2D Coulomb gas : n particles in $\mathbb{R}^2$ with coulombic pairwise interaction + confining potential (with E. Sandier)
2. vortices in the Ginzburg-Landau model of superconductivity (with E. Sandier)
3. droplets in the Ohta-Kawasaki model (with D. Goldman and C. Muratov)

Outline

I. Presentation of the three models

II. Details for the Coulomb gas model (case of energy minimizers)

III. Analogous results for the other two models

IV. Application of II to the statistical mechanics of the the Coulomb gas

V. Extensions

Energy of a 2D Coulomb gas in a potential V / "weighted Fekete sets"

$$w_n(x_1, \cdots, x_n) = -\sum_{i \neq j} \log |x_i - x_j| + n \sum_{i=1}^{n} V(x_i) \qquad x_i \in \mathbb{R}^2$$

Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n \frac{V}{2}(x_i)}$$

$\to$ **weighted Fekete sets** (interpolation).

Note: choosing V appropriately and using stereographic projection, reduces to the question of Fekete points on the sphere ($\max_{x_i \in \mathbb{S}^2} \prod_{i<j} |x_i - x_j|$).

Also related to some random matrix models (see later)

Limit $n \to \infty$?

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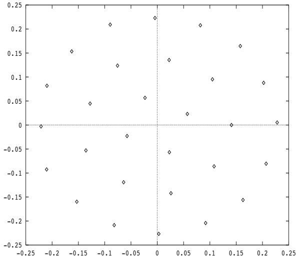


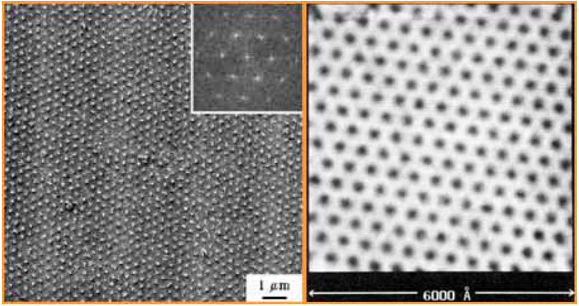
Numerical minimization of w_n **for** $V(x) = |x|^2$ (Gueron-Shafrir), $n = 29$

Vortices in the Ginzburg-Landau model of superconductivity

$$G_\varepsilon(\psi, A) = \frac{1}{2}\int_\Omega |(\nabla - iA)\psi|^2 + |\nabla \times A - h_{\rm ex}|^2 + \frac{(1-|\psi|^2)^2}{2\varepsilon^2}.$$

- $\Omega \subset \mathbb{R}^2$
- $\psi : \Omega \to \mathbb{C}$ "order parameter"
- $|\psi|^2$= density of superconducting Cooper pairs. $\psi = 0$ **vortices**
- $A : \Omega \to \mathbb{R}^2$
- $h_{\rm ex} > 0$ intensity of applied field
- limit $\varepsilon \to 0$ (material constant)

For $H_{c_1} < h_{\rm ex} \ll \frac{1}{\varepsilon^2}$, minimizers (ψ, A) of G_ε have vortices which form triangular "Abrikosov" lattices



Abrikosov lattice

The Ohta-Kawasaki model of "diblock copolymers"

$$E_\varepsilon(u) = \varepsilon \int_{\mathbb{T}} |\nabla u| + \int_{\mathbb{T}\times\mathbb{T}} G(x-y)(u-\bar{u})(x)(u-\bar{u})(y)\,dx\,dy$$

- $u : \mathbb{T} = [0,1]^2 \to \{-1,1\}$
- $\bar{u} = \fint_{\mathbb{T}} u$ prescribed
- G Green's function for $-\Delta + \kappa^2 I$.
- Two phases $+1$ and -1 interacting via a screened Coulomb kernel. $\int |\nabla u|$ is the total perimeter of the interfaces. It competes with the repulsive term $\int_{\mathbb{T}\times\mathbb{T}} G(x,y)(u-\bar{u})(x)(u-\bar{u})(y)\,dx\,dy$ which prefers rapid oscillation between the phases.
- Many possible regimes. Choksi-Peletier, Alberti-Choksi-Otto, Muratov, Ren... Structure of minimizers expected to be periodic but not proven!

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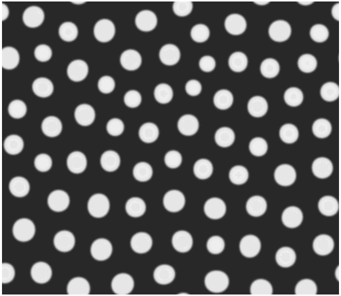

Numerical minimization (C. Muratov)

- Interested in regime in which one phase is in strong majority compared to the other $\rightsquigarrow$ almost round "droplets" of phase -1 in a sea of $+1$, asymptotically points, number diverges as $\varepsilon \to 0$

Traits in common

In the studied regime the three models mathematically boil down to the same

- Coulomb 2D interaction
- number of points diverging
- pattern formation in minimizers: periodic, even triangular lattice!

Coulomb gas and equilibrium measure

$$w_n(x_1, \cdots, x_n) = -\sum_{i \neq j} \log|x_i - x_j| + n \sum_{i=1}^{n} V(x_i) \qquad x_i \in \mathbb{R}^2$$

V is regular enough and assumed to grow faster than $\log|x|$ at infinity.
Define

$$\mathcal{F}(\mu) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} -\log|x-y| \, d\mu(x) \, d\mu(y) + \int_{\mathbb{R}^2} V(x) \, d\mu(x).$$

$\mathcal{F}$ has a unique minimizer among probability measures, called the *equilibrium measure*, denoted μ_0. Frostman

Denote $E = Supp(\mu_0)$ (assumed to be compact with C^1 boundary).

Example: $V(x) = |x|^2$, then $\mu_0 = \frac{1}{\sqrt{\pi}} \mathbb{1}_{B_1}$ (circle law).

Proposition (mean field limit)

$$\frac{w_n}{n^2} \ \Gamma - \textit{converges to } \mathcal{F}:$$

$$\lim_{n \to \infty} \frac{\min w_n}{n^2} = \mathcal{F}(\mu_0) \qquad \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \delta_{x_i}}{n} = \mu_0 \quad \textit{for a minimizer}$$

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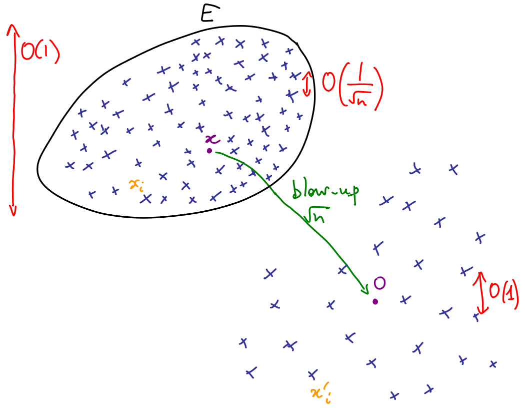
Objective

We know the global distribution of the points is μ_0 and $\min w_n \sim n^2 \mathcal{F}(\mu_0)$.
Can we say more about the local distribution of points and the next order terms in $\min w_n$?? For that we want to blow up the points at the scale $\sqrt{n}$ to see them at finite distances from each other.



Splitting of w_n

The idea is to understand the next order behavior by splitting w_n, writing

$$\nu_n := \sum_{i=1}^{n} \delta_{x_i}$$

as $n\mu_0 + (\nu_n - n\mu_0)$.

$$w_n(x_1, \cdots, x_n) = \iint_{\triangle^c} -\log|x-y| \, d\underbrace{\nu_n(x)}_{n\mu_0+(\nu_n-n\mu_0)} \, d\underbrace{\nu_n(y)}_{n\mu_0+(\nu_n-n\mu_0)} + \int V(x) \, d\underbrace{\nu_n(x)}_{n\mu_0+(\nu_n-n\mu_0)} .$$

[Reminder $w_n(x_1, \cdots, x_n) := -\sum_{i \neq j} \log|x_i - x_j| + n\sum_{i=1}^{n} V(x_i)$]

We find

$$w_n(x_1,\cdots,x_n) = n^2\mathcal{F}(\mu_0) + 2n\sum_{i=1}^n \zeta(x_i) + \underbrace{\frac{1}{\pi}W(\nabla H_n, \mathbb{1}_{\mathbb{R}^2})}_{\text{“renormalized" self-interaction of } \nu_n - n\mu_0}$$

Here

$$\begin{cases} \zeta = cst + \frac{1}{2}V - \int \log|x-y|\, d\mu_0(y) & \\ \zeta = 0 & \text{in } E \\ \zeta > 0 & \text{in } \mathbb{R}^2\setminus E \end{cases}$$

$$H_n = -2\pi\Delta^{-1}\left(\sum_{i=1}^n \delta_{x_i} - n\mu_0\right) = -\log|x| * \left(\sum_{i=1}^n \delta_{x_i} - n\mu_0\right)$$

and for every function χ,

$$W(\nabla H_n, \chi) := \lim_{\eta\to 0}\int_{\mathbb{R}^2\setminus\cup_{i=1}^n B(x_i,\eta)} \chi|\nabla H_n|^2 + \pi(\log\eta)\sum_i \chi(x_i).$$

In rescaled coordinates $x' = \sqrt{n}(x - x_0)$ this becomes

$$w_n(x_1, \cdots, x_n) = n^2 \mathcal{F}(\mu_0) - n \log \sqrt{n} + \frac{1}{\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^{n} \zeta(x_i)$$

where H'_n is the solution to

$$H'_n(x') = -2\pi \Delta^{-1} \left(\sum_{i=1}^{n} \delta_{x'_i} - \mu_0 (x_0 + \frac{x'}{\sqrt{n}}) \right)$$

- in the limit $n \to \infty$

$$-\Delta H' = 2\pi \left(\sum_i \delta_{x'_i} - \mu_0(x_0) \right)$$

- remains to understand $W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2})$, "renormalized" Coulomb interaction between the points in a neutralizing background, of slowly varying density $\sim \mu_0$
- difficulties in letting $n \to \infty$, in particular no local "charge neutrality"
- need to define a total Coulomb interaction for such a system with infinite number of points

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Complete definition of W

Let $m > 0$ be given. We say a vector field j belongs to the class $\mathcal{A}_m$ if

$$j = \nabla H \quad -\Delta H = 2\pi(\nu - m)$$

for some $\nu = \sum_{p \in \Lambda} \delta_p$ where Λ is a discrete set.

Definition

For $j \in \mathcal{A}_m$, for any smooth positive χ, let

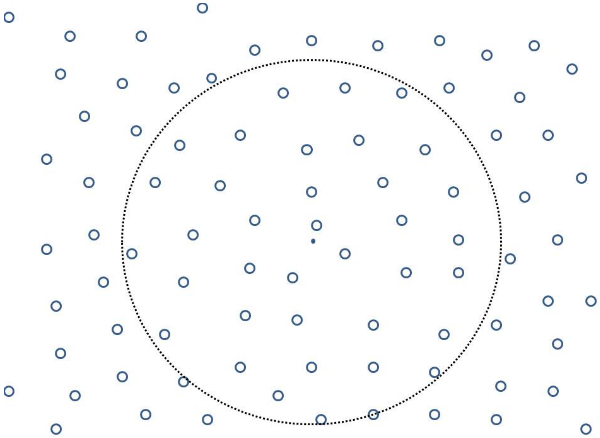
$$W(j,\chi) = \lim_{\eta \to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p,\eta)} \chi |j|^2 + \pi \log \eta \sum_{p \in \Lambda} \chi(p) \right).$$

We define the **renormalized energy** W *by*

$$W(j) := \limsup_{R \to \infty} \frac{W(j, \chi_{B_R})}{|B_R|},$$

where χ_{B_R} is any cutoff function supported in B_R with $\chi_{B_R} = 1$ in B_{R-1} and $|\nabla \chi_{B_R}| \leq C$.

Computing W



The case of the torus

Assume Λ is $\mathbb{T}$-periodic. Then W can be written as a function of Λ "=" $\{a_1, \ldots, a_n\}$.

$$W(a_1, \cdots, a_n) = \frac{\pi}{|\mathbb{T}|} \sum_{j \neq k} G(a_j - a_k) + \pi \lim_{x \to 0} \left(G(x) + \log |x| \right),$$

where G= Green's function of the torus ($-\Delta G = \delta_0 - 1/|\mathbb{T}|$).

[On a square torus

$$W(a_1, \cdots, a_n) = \frac{1}{2n} \sum_{j \neq k} E(a_j - a_k) + \pi \log \frac{\sqrt{n}}{2\pi} - 2\pi \log \eta(i)$$

Here $E(x) = E_{\Re(x/N), \Im(x/N)}(i)$ where $E_{u,v}(\tau)$ is the Eisenstein series defined for $\tau \in \mathbb{C}$ and $u, v \in \mathbb{R}$ by

$$E_{u,v}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} e^{2i\pi(mu+nv)} \frac{\Im(\tau)}{|m\tau + n|^2}.$$

and η denotes the Dedekind η function $\eta(\tau) = (e^{2i\pi\tau})^{1/24} \prod_{k=1}^{\infty} (1 - (e^{2i\pi\tau})^k)$.]

Such quantities also arise on general Riemann surfaces in Arakelov theory (cf. Lang).

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$$W(j) = m\left(W(j') - \frac{\pi}{2}\log m\right)$$

  so we can reduce to $\mathcal{A}_1$.
- W is unchanged by a compact perturbation of the point configuration
- Proposition: Minimizers of W exist (requires work, lower semi-continuity up to translations)
- Proposition: $\min_{\mathcal{A}_1} W$ is the limit as $N \to \infty$ of the min over $\mathbb{T}_N$-periodic configurations.

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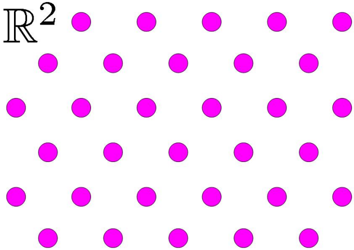
Minimization among lattices

We can look for minimizers of W among perfect lattice configurations, i.e., $\Lambda = \mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$, with unit volume.

Theorem (Sandier-S. CMP'12)

The minimum of $\Lambda \mapsto W(\Lambda)$ over perfect lattice configurations is achieved uniquely, modulo rotations, by the triangular lattice.



- use explicit formula in terms of Eisenstein series
- by transformations using modular functions or by direct computations, minimizing W becomes equivalent to minimizing the Epstein zeta function $\zeta(s) = \sum_{p\in\Lambda} \frac{1}{|p|^s}$, $s > 2$, over lattices
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A crystallization question

- the question belongs to the more general family of crystallization problems: given V, what are the point positions that minimize

$$\sum_{i\neq j} V(x_i - x_j)$$

(some kind of boundary condition needed)? Or rather

$$\lim_{R\to\infty} \frac{1}{|B_R|} \sum_{i\neq j, x_i, x_j \in B_R} V(x_i - x_j)?$$

Are they perfect lattices?
cf. cristalline structure of matter, Fekete points, "Smale's problem" on the sphere, the "Cohn-Kumar conjecture"

- very few positive results in the literature
- here the potential is very "long range"

- for 2D sphere packing (hard spheres potential) Radin proves the triangular lattice is a minimizer
- for a very short range Lennard-Jones potential Theil proves the same result
- the question here

$$\min \text{“}\| \sum_i \delta_{x_i} - 1 \|_{(W^{1,2})^*}\text{”}$$

- $$\min \| \sum_i \delta_{x_i} - 1 \|_{Lip^*}$$

  achieved by triangular lattice (Bourne-Peletier-Theil, also for the r-Wasserstein distance)
- W is expected to be a quantitative "measure of disorder" for a configuration of points in the plane.

Results for minimizers of w_n

Theorem (Sandier-S. arXiv '12)

Let $(x_1, \dots, x_n)$ *and* $\nu_n = \sum_{i=1}^n \delta_{x_i}$. *Then*

$$\liminf_{n\to\infty} \frac{1}{n}\left(w_n(x_1,\dots,x_n) - n^2\mathcal{F}(\mu_0) + \frac{n}{2}\log n\right) \geq \frac{|E|}{\pi}\int W(j)\,dP(x,j),$$

where P − *a.e.*$(x.j) \in \mathcal{A}_{\mu_0(x)}$, *and* P *is a probability, limit of the push-forward of* $\frac{1}{|E|}dx_{|E}$ *by*

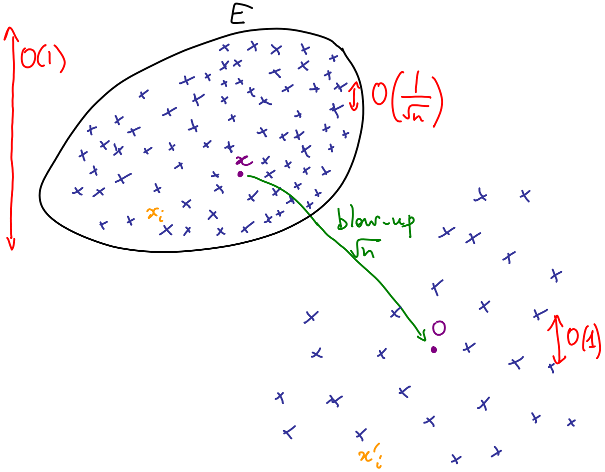
$$x \mapsto \left(x, j_n(\sqrt{n}x + \cdot)\right), \qquad j_n := \nabla H_n'.$$

This lower bound is sharp; thus for minimizers $P-$*a.e.* j *minimizes* W *over* $\mathcal{A}_{\mu_0(x)}$ *and*

$$\lim_{n\to\infty} \min \frac{1}{n}\left(w_n - n^2\mathcal{F}(\mu_0) + \frac{n}{2}\log n\right) = \frac{|E|}{\pi}\int\left(\min_{j\in\mathcal{A}_{\mu_0(x)}} W\right)dP(x,j)$$

$$= \frac{1}{\pi}\min_{\mathcal{A}_1} W - \frac{1}{2}\int \mu_0\log\mu_0 := \alpha_0.$$

Heuristic rephrasing: If $(x_1, \cdots, x_n)$ minimize w_n, after blow-up at scale $\sqrt{n}$ around a point x chosen "uniformly at random" in E, the limit of $\nabla H_n'$ minimizes W over $\mathcal{A}_{\mu_0(x)}$.



E
O(1)
$O\left(\frac{1}{\sqrt{n}}\right)$
x
x_i
blow-up
$\sqrt{n}$
0
O(1)
x'_i

Extensions

Work in progress with Rota Nodari:

If $(x_1, \ldots, x_n)$ minimizes w_n then

- for *every* blow up center x, the limiting blown-up j minimizes W over $\mathcal{A}_{\mu_0(x)}$
- for *every* x, the number of points in $B(x, \frac{R}{\sqrt{n}})$ is $\pi R^2 \mu_0(x) + o(R^2)$ as $n \to \infty$ then $R \to \infty$.

Second result proven by Ameur - Ortega Cerda '11 by the method of "Beurling-Landau densities".

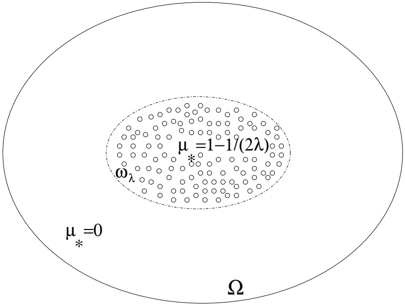
Similar result for Ginzburg-Landau

Mean field description for $h_{\rm ex} > H_{c_1}$ (Sandier-S, Annales ENS '00)

$$h_{\rm ex} = \lambda|\log\varepsilon|, \quad \lambda > \lambda_\Omega$$

μ_ε density of vortices (weighted by their degrees)

$$\frac{\mu_\varepsilon}{h_{\rm ex}} \rightharpoonup \mu_*$$



μ_* **solution of a free boundary (obstacle) problem**

Theorem (Sandier-S, CMP'12)

Consider minimizers $(u_\varepsilon, A_\varepsilon)$ *of the Ginzburg-Landau in the regime* $\lambda_\Omega |\log \varepsilon| \le h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}$.
After blow-up around a randomly chosen point in ω_λ*, their "currents"* $\nabla h_\varepsilon (= \nabla \mathrm{curl}\, A_\varepsilon)$ *converge as* $\varepsilon \to 0$ *to currents in the plane which, almost surely, minimize* W.
Moreover, $\min G_\varepsilon$ *can be computed up to* $o(h_{\mathrm{ex}})$.

- Compare to Bethuel-Brezis-Hélein, S, Sandier-S, for bounded number of vortices
- If conjecture on $\min W$ is true then this would completely justify the emergence of the Abrikosov lattices in superconductors.

Similar results for Ohta-Kawasaki

$$E_\varepsilon(u) = \varepsilon \int_{\mathbb{T}} |\nabla u| + \int_{\mathbb{T}\times\mathbb{T}} G(x-y)(u-\bar{u})(x)(u-\bar{u})(y)\, dx\, dy$$

We can write $u = -1 + 2\sum_i \chi_{\Omega_i}$, Ω_i=droplets of phase $+1$.

Theorem (Goldman-Muratov-S, to appear)

For energy minimizers (or almost minimizers), most droplets rescaled by $\varepsilon^{-1/3}|\log \varepsilon|^{1/3}$ *converge in Hausdorff distance to round droplets of fixed radius* $3^{1/3}$*, and "minimizing"* $\int W\, dP$ *after blow up at scale* $\sqrt{|\log \varepsilon|}$.

⇝ Again expect the triangular (Abrikosov) lattice in this regime of very many very small droplets!

Method of the proof

- Γ-convergence: prove general (ansatz-free) lower bounds and upper bounds which match
- Introduce a new general method for lower bound on two-scale energies (after splitting + blow-up, the domain becomes of infinite size, so it is difficult to localize energy lower bounds). A probability measure approach allows to do this via the use of the ergodic theorem (idea of Varadhan)
- That method applies well to positive (or bounded below) energy densities, but those associated to $W(\nabla H, \chi)$ are not!
- difficulty: lack of local charge neutrality
- Start by modifying the energy density to make it bounded below: method of mass transport, using sharp energy lower bounds by "ball construction" methods (à la Jerrard / Sandier)
- For GL same as for Coulomb gases, but complicated by the presence of vortices of arbitrary signs and degrees
- For Ohta-Kawasaki, we have the constant sign of the charges, but their mass (= volume of the droplets) is not quantized, and their shape is not fixed a priori, saved by the isoperimetric inequality

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Beyond minimizers: the statistical mechanics of the Coulomb gas

Consider the probability law

$$d\mathbb{P}_{n,\beta}(x_1, \cdots, x_n) = \frac{1}{Z_{n,\beta}} e^{-\beta w_n(x_1, \cdots, x_n)} dx_1 \cdots dx_n$$

where $Z_{n,\beta}$ is the associated partition function, and

$$w_n(x_1, \cdots, x_n) = -\sum_{i \neq j} \log |x_i - x_j| + n \sum_{i=1}^{n} V(x_i).$$

and $x_i \in \mathbb{R}^d$ with $d = 1$ or 2.
For general β and V, these ensembles are called log gases in dimension 1 and Coulomb gases in dimension 2. Some important and well-studied examples are **random matrix models** (first noticed by Dyson):

- For $d = 1$, $\beta = 2$, $V(x) = x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d = 1$, $\beta = 1$, $V(x) = x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d = 2$, $\beta = 2$ and $V(x) = |x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).

Statistical mechanics approach: 3D Lieb-Narnhofer, Lieb-Oxford (3D) Alastuey-Jancovici, Jancovici-Leibowitz-Manificat, Sari-Merlini, Frölich-Spencer, Kiessling-Spohn, Zabrodin-Wiegmann...

Random matrix texts Anderson-Guionnet-Zeitouni, Deift, Forrester, Mehta.

Next-order expansion of the partition function ($d = 2$)

Theorem (Sandier-S. arXiv '12)

$$n\beta f_1(\beta) \le \log Z_{n,\beta} - \Big(-\beta n^2 \mathcal{F}(\mu_0) + \frac{\beta n}{2}\log n \Big) \le n\beta f_2(\beta),$$

where $f_1(\beta)$ and $f_2(\beta)$ are independent of n, bounded, and

$$\lim_{\beta\to\infty} f_1(\beta) = \lim_{\beta\to\infty} f_2(\beta) = \alpha_0,$$

where

$$\alpha_0 = \frac{1}{\pi}\min_{\mathcal{A}_1} W - \frac{1}{2}\int \mu_0 \log \mu_0 \, dx.$$

Not previously known in dimension 2 (no Selberg integrals formulae), relates the computation of $Z_{n,\beta}$ to that of the unknown constant $\min_{\mathcal{A}_1} W$.

A large deviations result

Theorem (Ben Arous-Guionnet $d = 1$, Ben Arous-Zeitouni $d = 2$, Hiai-Petz)

$\mathbb{P}_{n,\beta}$ *satisfies a large deviations principle with good rate function* $\mathcal{F}(\cdot)$ *and speed* n^{-2}*: for all* $A \subset \{\text{probability measures}\}$,

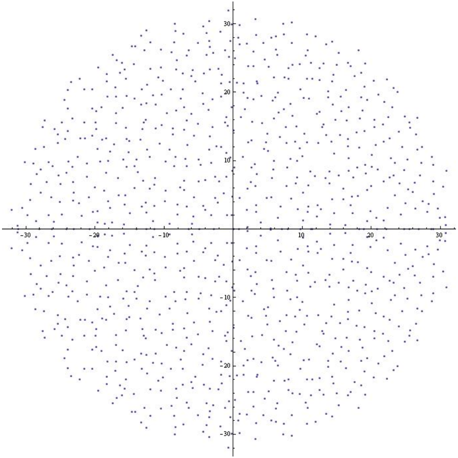
$$-\inf_{\mu \in A^\circ} \widetilde{\mathcal{F}}(\mu) \le \liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_{n,\beta}(A) \le \limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_{n,\beta}(A) \le -\inf_{\mu \in \bar{A}} \widetilde{\mathcal{F}}(\mu),$$

where $\widetilde{\mathcal{F}} = \mathcal{F} - \min \mathcal{F}$.

[Recall $\mathcal{F}(\mu) = \iint \log|x-y| \, d\mu(x) \, d\mu(y) + \int V(x) \, d\mu(x)$, uniquely minimized by μ_0]

$$\mathbb{P}_{n,\beta}(A) \le e^{-n^2 \inf_{\bar{A}} (\mathcal{F} - \mathcal{F}(\mu_0))}.$$

A simulation



Eigenvalues of 1000-by-1000 matrix with i.i.d Gaussian entries ($\beta = 2$, $\mu_0 = \frac{1}{\sqrt{\pi}} \mathbb{1}_{B_1}$) (Borrowed from Benedek Valkó's webpage)

"Large deviations type" result at next order

Theorem (Sandier-S. arXiv '12)

Let $A_n \subset (\mathbb{R}^2)^n$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{n,\beta}(A_n) \le -\beta\Big(\underbrace{\frac{|E|}{\pi} \inf_{P\in A} \int W(j)\,dP(x,j)}_{\text{min of this is } \alpha_0} - \alpha_0 - \frac{C}{\beta}\Big),$$

and A is the set of probability measures which are limits of blow-ups at rate $\sqrt{n}$ around a point x of the current j associated to $\sum_{i=1}^n \delta_{x_i}$ with $(x_i) \in A_n$.

- For β finite, the average of W lies below a fixed constant $(\alpha_0 + \frac{C}{\beta})$, except with very small probability.
- modulo the conjecture, this proves **crystallization** as $\beta \to \infty$: $\rightsquigarrow$ after blowing up around a point x in the support of μ_0, at the scale of $(n\mu_0(x))^{1/2}$, we see (almost surely) a configuration which minimizes W.

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Extensions

- extension of the definition of W to 1D (with E. Sandier)

$$-\Delta H = 2\pi(\sum_i \delta_{x_i} - \mu_0) \quad \text{in } \mathbb{R}^2$$

  where x_i are points on the real axis of $\mathbb{R}^2$, μ_0 measure supported on the real axis of $\mathbb{R}^2$.
- analogous results should hold for minimizers of w_n, for statistical mechanics of the log gas (calculation of $Z_{n,\beta}$, large deviations etc)
- In 1D, $\min W$ is achieved by the perfect lattice $\mathbb{Z}$, and the crystallisation result should be complete!
- use W to quantify the disorder of some classic random point configurations in the plane and on the real line (with A. Borodin)
- usual Fekete points on a compact set (with A. Contreras and E. Sandier)
- quantum Coulomb gases in 2D (with M. Lewin, P. T. Nam, and J. P. Solovej).
- the limit $\beta \to 0$ (with N. Rougerie)

THANK YOU FOR YOUR ATTENTION!