On neighborhoods of analytic functions

Janusz Sokół
Department of Mathematics
Rzeszów University of Technology
ul. Powsańców Warszawy 12, 35-959 Rzeszów, Poland
e-mail: jsokol@prz.edu.pl

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Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. Everywhere in this abstract $z \in \mathcal{U}$. The T_δ-neighborhood ($\delta > 0$) of the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is defined as

$$TN_\delta(f) = \{g(z) = z + \textstyle\sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \le \delta\},$$

where T is a sequence $T = \{T_n\}_{n=2}^{\infty}$ consisting of positive numbers. If $T = \{n\}_{n=2}^{\infty}$ then T_δ-neighborhood becomes the δ-neighborhood $N_\delta(f)$ introduced by St. Ruscheweyh who used it to generalize the earlier result due to A. W. Goodman that $N_1(z) \subseteq \mathcal{S}^*$ by showing that if $f \in \mathcal{C}$ then $N_{1/4}(f) \subseteq \mathcal{S}^*$, where $\mathcal{C}$, $\mathcal{S}^*$ denote the well known classes of convex and starlike functions, respectively. Some results of this type one can find also in papers of R. Founier, J. Stankiewicz. The TN_δ-neighborhood was introduced by T. Sheil-Small and E.M. Silvia, who considered the problem of finding a sufficient condition on $f \in \mathcal{A}$ that implies the existence of $TN_\delta(f)$ being contained in a given subclass. They proved a number of theorems showing the importance of convolutions in the study of TN_δ-neighborhoods. If a function f have an another normalization in the origin, then it seems to be rightly to consider a modified δ-neighborhoods. In this work we will consider a δ-neighborhood of $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, denoted by $\widehat{N}_\delta(p)$, defined by $\widehat{N}_\delta(p) = \{q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n : \sum_{n=1}^{\infty} |p_n - q_n| \le \delta\}$. We apply the following result.

Theorem 1. *Let $\kappa < 1$, $\zeta \in \mathbb{C}$ and $\Re\zeta > \kappa$. If $h \in \mathcal{A}$ and $\zeta h'(z) \in \Omega(\kappa, \zeta)$, where $\Omega(\kappa,\zeta) = \left\{w \in \mathbb{C} : \left|w - 2\kappa + \overline{\zeta}\right| > |\Re w - \kappa|\right\}$, then*

$$\Re\left[\zeta \frac{h(z)}{z}\right] > \kappa \quad (z \in \mathcal{U}).$$

The last κ can not be replaced by a greater number.

It is easy to see that for $\beta < 1$ and $\zeta \in \mathbb{C}$ the set $\Omega(\beta, \zeta)$ is the set of all points in the plane such that the distance from the focus $2\beta - \overline{\zeta}$ is greater then the distance from the directrix $\Re w = \beta$, so $\Omega(\beta,\zeta)$ is a concave set laying on the right side of the parabola $\gamma : \left|w - 2\beta + \overline{\zeta}\right| = |\Re w - \beta|$ when $\beta < \Re\zeta$ while on the left one when $\beta > \Re\zeta$.

$\gamma : \left| w - 2\beta + \overline{\zeta} \right| = |\Re w - \beta|$

$\beta = \frac{1}{2}$; $\zeta = 1 - i$

directrix $l : \Re w = \frac{1}{2}$; focus $F = i$

$A = \frac{1}{4} + i$,

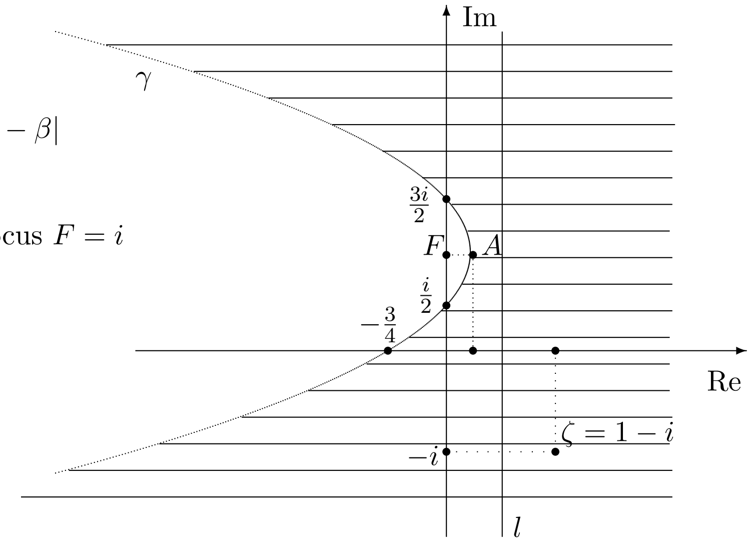

Fig.1. $\Omega(\frac{1}{2}, 1 - i)$.

We shall focus on a class of functions directly related to Theorem 1. Let us consider two classes of functions for $\alpha < 1$:

$$\mathcal{P}(\alpha) = \{p : zp(z) \in \mathcal{A} \text{ and } \Re\,[p(z)] > \alpha \text{ for } z \in \mathcal{U}\}$$

and a related class

$$\mathcal{P}'(\alpha) = \{p : zp(z) \in \mathcal{A} \text{ and } [zp(z)]' \in \Omega(\alpha) \text{ for } z \in \mathcal{U}\}\,,$$

where $\Omega(\alpha) = \Omega(\alpha, 1) = \{w \in \mathbb{C} : |w - 2\alpha + 1| > |\Re w - \alpha|\}\,.$

In this work we consider for given $\beta \leq \alpha < 1$ and $q \in \mathcal{P}'(\alpha)$ the problem of finding a sufficient condition on δ that implies the existence of $\widehat{N}_\delta(q)$ being contained in $\mathcal{P}(\beta)$.

Theorem 2. *If $\alpha \leq \beta < 1$ and $q \in \mathcal{P}'(\alpha)$, then*

$$\widehat{N}_\delta(q) \subseteq \mathcal{P}(\beta)$$

whenever $\delta \leq \alpha - \beta$.