The generalized Newton–Kantorovich method for equations with nondifferentiable operators

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Let X and Y be Banach spaces, D is a convex subset of X, f and g are nonlinear operators, defined on D and taking values in Y, where f is differentiable at every interior point of D, g is nondifferentiable. The generalized Newton–Kantorovich method for solving operator equations of the form

$$f(x) + g(x) = 0 \tag{1}$$

is studied. For analyzing successive approximations

$$x_{n+1} = x_n - [f'(x_n)]^{-1}(f(x_n) + g(x_n)) \quad (n = 0,\ 1,\ \ldots), \tag{2}$$

where $x_0 \in D$ is given, the approach based on application of majorant equations and originating from Kantorovich's investigations [1] is used.

In the case when $g = 0$ the most precise error estimates for the process (2) were obtained in [2, 3] under a new smoothness assumption imposed on the operator f called regular smoothness. We generalize the main result from [3] to equations of the form (1) under the hypotheses that the operator f is regularly smooth on D and the operator g satisfies the condition

$$\|g(x'') - g(x')\| \le \psi(t)\|x'' - x'\|, \quad \forall\, x',\ x'' \in \overline{B(x_0, t)} \subseteq D,$$

where $\psi(t)$ is the nondecreasing function of the nonnegative argument.

Let $\mathcal{N}$ denote the class of continuous strictly increasing functions $\omega : [0,\infty) \to [0,\infty)$ that are concave and vanish at zero: $\omega(0) = 0$. Denote by $h(f)$ the quantity $\inf\limits_{x\in D} \|f'(x)\|$. Given an $\omega \in \mathcal{N}$, we say in accordance with [3] that f is *ω-regularly smooth on D* if exists an $h \in [0, h(f)]$ such that the inequality

$$\omega^{-1}\left(h_f(x', x'') + \|f'(x'') - f'(x')\|\right) - \omega^{-1}\left(h_f(x', x'')\right) \le \|x'' - x'\|,$$

where $h_f(x', x'') = \min\{\|f'(x')\|,\ \|f'(x'')\|\} - h$, holds for all $x',\ x'' \in D$.

The operator f is called *regularly smooth* on D, if it is ω-regularly smooth on D for some $\omega \in \mathcal{N}$.

Assume without loss of generality that $f'(x_0) = I$. Denote

$$\Omega(t) = \int_0^t \omega(\tau)\, d\tau, \qquad \Psi(t) = \int_0^t \psi(\tau)\, d\tau, \qquad \chi = \omega^{-1}(1 - h),$$

a — positive number such that $\|f(x_0) + g(x_0)\| \le a$ and

$$W(t) = a - \Omega(\chi) + \Omega(\chi - t) - th + \Psi(t), \qquad t \in [0,\ \chi]. \tag{3}$$

Theorem. Suppose that the function (3) has a unique zero t_* in the interval $[0, \chi]$, the closed ball $\overline{B(x_0, t_*)}$ is contained in D and $a < \Omega(\chi) + h \cdot \chi - \Psi(\chi)$. Then

1) the equation (1) has a unique solution x_* in the ball $\overline{B(x_0, t_*)}$;

2) the successive approximations (2) are defined for all $n = 0,\ 1,\ \ldots$, belong to $\overline{B(x_0, t_*)}$ and converge to x_*;

3) for all $n = 0,\ 1,\ \ldots$ the inequalities

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \qquad \|x_* - x_n\| \le t_* - t_n,$$

hold, where the sequence $\{t_n\}$ is defined by the recurrence formula

$$t_{n+1} = t_n + \frac{W(t_n)}{h + \omega(\chi - t_n)} \qquad (n = 0,\ 1,\ \ldots;\ t_0 = 0),$$

monotonically increases and converges to t_*.

References

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