ON SUFFICIENT CONDITION FOR STRONGLY STARLIKENESS

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Let $\mathcal{A}$ denote the class of functions with the series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

in the unit disc $U = \{z : |z| < 1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{S}$ is said to be *starlike of order* α if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U),$$

for some $0 \le \alpha < 1$, Robertson [6]. We denote by $\mathcal{S}^*(\alpha)$ the class of functions starlike of order α. We say that a function $f \in \mathcal{S}$ is *strongly starlike of order* β if and only if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \quad (z \in U),$$

for some β $(0 < \beta \le 1)$. Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β. The class $\mathcal{SS}^*(\beta)$ was introduced independently by Stankiewicz [8, 9] and by Brannan and Kirvan [1]. In [10] Takahashi and Nunokawa defined the following subclass of $\mathcal{A}$:

$$\mathcal{S}^*(\alpha,\beta) = \left\{f \in \mathcal{A} : \frac{-\pi\beta}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\pi\alpha}{2},\ z \in U\right\},$$

for some $0 < \alpha \le 1$, and for some $0 < \beta \le 1$. We recall here the fact, that in [2] and in [3] a similar class was studied. Note that $\mathcal{SS}^*(\min\{\alpha,\beta\}) \subset \mathcal{S}^*(\alpha,\beta) \subset \mathcal{SS}^*(\max\{\alpha,\beta\})$. Of course for $\alpha = \beta$ the class $\mathcal{S}^*(\alpha,\beta)$ becomes the class $\mathcal{SS}^*(\beta)$. It is easily seen that $\mathcal{S}^*(\alpha,\beta) \subset \mathcal{S}^*$. In [7] Silverman examined the class $\mathcal{G}_b$ of mappings $f \in \mathcal{S}$ that satisfy the condition

$$\left|\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < b,\ z \in U,$$

for some positive b. In [7] the following inclusion result for the class $\mathcal{G}_b$ have been obtained :

Theorem 0.1. [7] *If* $0 < b \le 1$, *then*

$$\mathcal{G}_b \subset \mathcal{S}^*\left(\frac{2}{1+\sqrt{1+8b}}\right).$$

The result is sharp for all b.

In [4] the authors obtained :

Theorem 0.2. [4] *If* f *belongs to the class* $\mathcal{G}_{b(\beta)}$ *with*

$$b(\beta) = \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}},$$

then $f \in \mathcal{SS}^*(\beta)$.

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In this work we consider the analogous problem for the classes $\mathcal{G}_b$ and $\mathcal{S}^*(\alpha,\beta)$. Namely, given α,β we look for possible great b such that $\mathcal{G}_b \subset \mathcal{S}^*(\alpha,\beta)$.

Our main result is contained in:

Theorem 0.3. *Assume that* $0<\alpha\le 1$, $0<\beta\le 1$. *If* $f\in\mathcal{G}_{b(\alpha,\beta)}$ *with*

$$b(\alpha,\beta)=\min\left\{\frac{\delta(\widetilde{x}_1^{1-\delta}-2\widetilde{x}_1^{-\delta}\sin\theta+\widetilde{x}_1^{-1-\delta})}{2\cos\theta},\frac{\delta(\widetilde{x}_2^{1-\delta}+2\widetilde{x}_2^{-\delta}\sin\theta+\widetilde{x}_2^{-1-\delta})}{2\cos\theta}\right\},$$

where

$$\delta=\frac{\alpha+\beta}{2},\quad \theta=\frac{\pi}{2}\left(\frac{\alpha-\beta}{\alpha+\beta}\right),$$

$$\widetilde{x}_1=\frac{\sqrt{1-\delta^2\cos^2\theta}-\delta\sin\theta}{1-\delta},\qquad \widetilde{x}_2=\frac{\sqrt{1-\delta^2\cos^2\theta}+\delta\sin\theta}{1-\delta},$$

then $f\in\mathcal{SS}^*(\alpha,\beta)$.

If we put $\alpha=\beta$ in the above theorem then we get the following corollary.

Corollary 0.4. *Assume that* $0<\alpha<1$. *If* $f\in\mathcal{G}_{b(\alpha)}$ *with*

$$b(\alpha)=\frac{\alpha}{2}\left\{\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{-\alpha-1}{2}}+\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{-\alpha+1}{2}}\right\}=\frac{\alpha}{\sqrt{(1-\alpha)^{1-\alpha}(1+\alpha)^{1+\alpha}}}$$

then $f\in\mathcal{SS}^*(\alpha)$.

This is the result from Theorem 0.2. Putting $\alpha=\frac{1}{2}$ in Corollary 0.4 we obtain

Corollary 0.5. *If*

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1\right|<\frac{\sqrt[4]{3}}{3},\ z\in U$$

then f *is strongly starlike of order* $\frac{1}{2}$.

Putting $\alpha=3/4$, $\beta=1/4$ in Theorem 0.3 we obtain

$$\delta=\frac{1}{2},\ \theta=\frac{\pi}{4},\ \widetilde{x}_1=\frac{\sqrt{2}(\sqrt{7}-1)}{2},\ \widetilde{x}_2=\frac{3\sqrt{2}(\sqrt{7}+1)}{2}$$

and

$$b(\alpha,\beta)=\min\left\{\frac{\sqrt[4]{2}(\sqrt{7}-2)}{3\sqrt{\sqrt{7}+1}},\ \frac{\sqrt[4]{2}(\sqrt{7}+5)}{6\sqrt{\sqrt{7}+1}}\right\}=\frac{\sqrt[4]{2}(\sqrt{7}-2)}{3\sqrt{\sqrt{7}+1}}.$$

Therefore, we may write the following corollary.

Corollary 0.6. *If* $f\in\mathcal{G}_{b(\alpha,\beta)}$ *with*

$$b(\alpha,\beta)=\frac{\sqrt[4]{2}(\sqrt{7}-2)}{3\sqrt{\sqrt{7}+1}}\approx 0.134,$$

then $f\in\mathcal{SS}^*(3/4,1/4)$.

Moreover, to obtain the main theorem we prove in this work the following version of the well known Jack's Lemma.

Theorem 0.7. *Let* p *be analytic in* U *with* $p(0)=1$ *and* $p(z)\neq 0$. *If there exist two points* $z_1\in U$ *and* $z_2\in U$ *such that* $|z_1|=|z_2|=r$ *and for* $z\in U_r=\{z:|z|<r\}$

$$-\frac{\pi\beta}{2}=\arg p(z_1)<\arg p(z)<\arg p(z_2)=\frac{\pi\alpha}{2},$$

with some $0 < \alpha \le 2$, $0 < \beta \le 2$, *then we have*

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\alpha+\beta}{2}m_1$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha+\beta}{2}m_2,$$

where

$$m_1 \ge \frac{1-t}{1+t}, \quad m_2 \ge \frac{1+t}{1-t},$$

and where

$$t = \tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right).$$

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