Tilings and Markov partitions

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Quasiperiodic motion = quasiperiodic flow

$$\begin{cases} \dot{x}_1 & = \omega_1 \\ \dot{x}_2 & = \omega_2 \end{cases} \qquad (x_1, x_2) \ \ mod\ 1$$

$\frac{\omega_2}{\omega_1}$ irrational $\Longrightarrow$ dense trajectories

$\frac{\omega_2}{\omega_1}$ rational $\Longrightarrow$ closed trajectories

Poincare section = rotation by $\frac{\omega_2}{\omega_1}$

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Another way to build quasiperiodic flow

suspension flow = suspension of interval exchange map

two rectangles K_1, K_2 of sizes $u \times p$ and $v \times q$

interval exchange map = rotation by $\frac{u}{u+v}$

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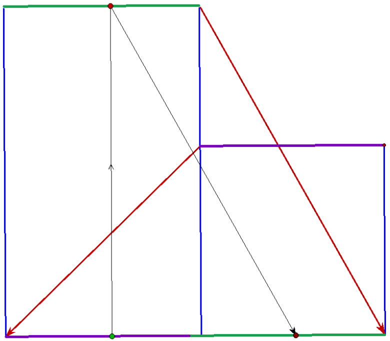

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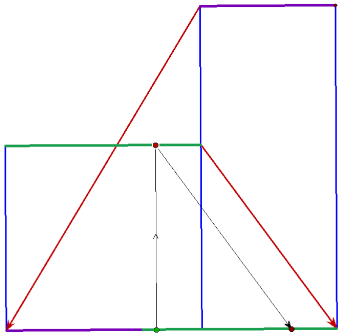

two translations generate lattice L

$$L = \left\{ \begin{bmatrix} v \\ p \end{bmatrix}, \begin{bmatrix} -u \\ q \end{bmatrix} \right\}$$

isomorphic to $\mathbb{Z}^2$

translating rectangles by vectors from the lattice $L \longrightarrow$
2-periodic tiling of $\mathbb{R}^2$

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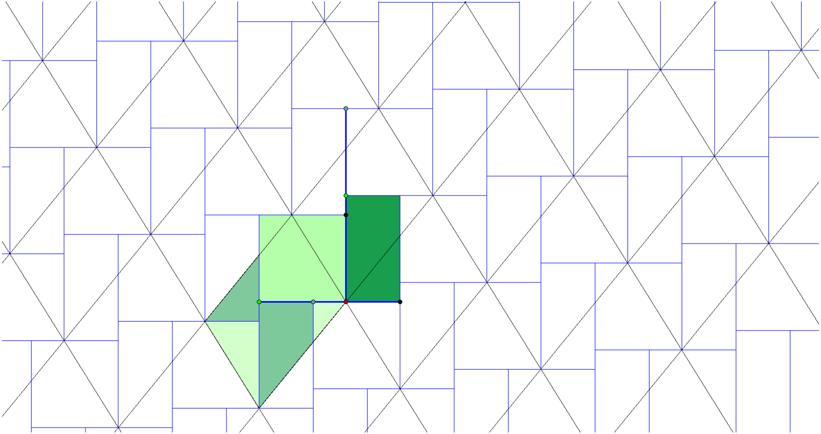
tiling generated by rectangles K_1, K_2 and lattice L



The torus

$$\mathbb{T}^2 = \mathbb{R}^2/L$$

natural projection

$$\pi : \mathbb{R}^2 \to \mathbb{R}^2/L = \mathbb{T}^2$$

union of the rectangles $K_1 \cup K_2 =$ fundamental domain of $\mathbb{T}^2$

partition into $\{K_1, K_2\}$ is called a *bi-partition* of $\mathbb{T}^2$
union of their horizontal sides J^s is called the *horizontal spine*,
union of the vertical sides J^u is called the *vertical spine*.

$$J^s \cap J^u = \{4 \text{ points}\}$$

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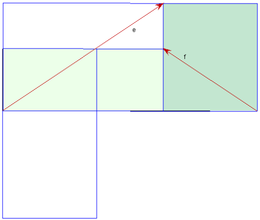
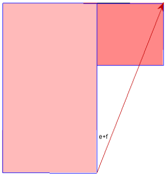
remodelling of bi-partition (Snavely '92)





bi-partition $\leftrightarrow$ basis (e, f) in lattice L

e in first quadrant, f in second quadrant

remodelled bi-partition $\longrightarrow$ modified basis

family $\mathcal{F}$ of such bases of L is called a *fan of bases*

cutting algorithm $\longrightarrow$ *cutting sequence* of Caroline Series

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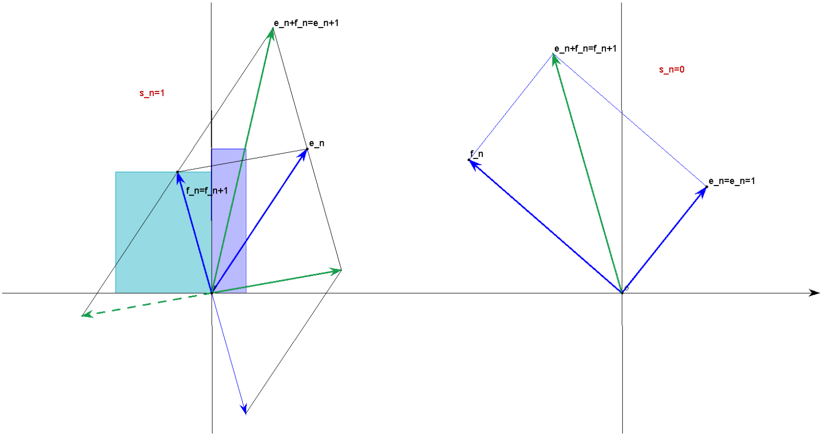

cutting algorithm

$(e_n, f_n) \longrightarrow g_n = e_n + f_n$

g_n is in the right half-plane $\longrightarrow e_{n+1} = g_n, f_{n+1} = f_n$

g_n is in the left half-plane $\longrightarrow e_{n+1} = e_n, f_{n+1} = g_n$

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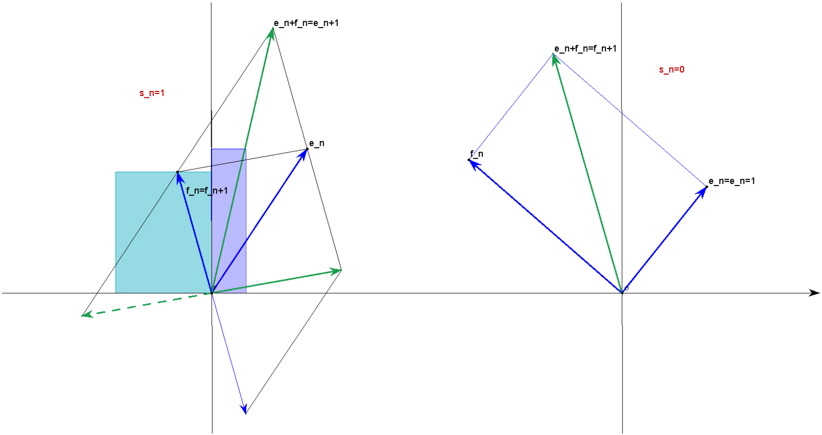

cutting algorithm exausts the fan of bases $\mathcal{F}$

$\mathcal{F}$ aquires order of $\mathbb{Z}$

fix $(e_0, f_0) \in \mathcal{F}$

fan $\mathcal{F}$ is completely described by *cutting sequence* $\{s_n\}$

for $n \in \mathbb{Z}$

$$s_n = 0 \quad \text{if} \quad e_{n+1} = e_n, \qquad s_n = 1 \quad \text{if} \quad f_{n+1} = f_n.$$

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cutting sequence $\underset{\textit{compression''}}{\overset{\textit{``data}}{\longrightarrow}}$ *geometric continued fraction*

cuttting sequence

$\ldots 0\,0\,0\,1\,1\,0\,1\,1\,1\,0\;.\;0\,0\,0\,1\,1\,1\,1\,1\,0\,0\,0\,0\,1\,0\,1\,1\,1\,1\,1\,1\,1 \ldots$

geometric continued fraction

$\ldots 3\,2\,1\,3\,1\;.\;3\,5\,4\,1\,1\,7 \ldots$

slopes of vertical and horizontal lines in basis (e_0, f_0)

$$\frac{v}{u} = 3 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \ldots}}}}} \qquad \frac{q}{p} = -\cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \ldots}}}}}$$

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Theorem (Lagrange)

The continued fraction of

$$\omega = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \dots}}} = [n_0; n_1, n_2, n_3, \dots]$$

is eventually periodic if and only if there are $a, b, c \in \mathbb{Z}$ s.t.

$$a\,\omega^2 + b\,\omega + c = 0.$$

Theorem

The cutting sequence is periodic if and only if the horizontal and vertical vectors are eigenvectors of an automorphism of the lattice L.

- very similar to a theorem of Caroline Series
- "theorem proves itself"
- (the group of automorphisms is isomorphic to $GL(2, \mathbb{Z})$)

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If an automorphism A of the lattice L preserves the horizontal and vertical lines then it preserves the fan of bases

$$\text{Fan of bases } = \mathcal{F} \ni (e, f) \longrightarrow (Ae, Af) \in \mathcal{F}$$

Hence the cutting sequence with respect to the basis (Ae, Af) is equal to the original and at the same time to the shifted one. □

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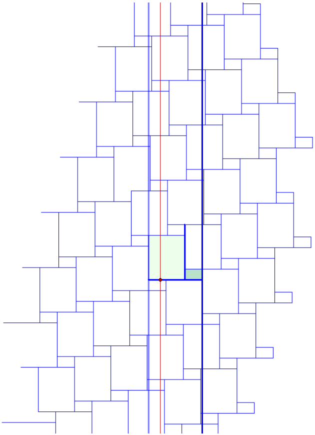

consider intersection tilings for all vertical lines

What kind of tilings arise in this way?

they are quasi-periodic (in some sense) = 1-dim quasi-crystals

1-dim tiling $\longrightarrow$ infinite word in 2 letters = bi-infinite sequence

bi-infinite sequence $\in \{p, q\}^{\mathbb{Z}}$

these are very special sequences!
= another kind of cutting sequences of Caroline Series

we will give 4 equivalent properties which define these intersection tilings

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1. *Sturmian* sequences:

$\forall n$ there are exactly $n+1$ words of length n

$n = 2,\ (p,q),(q,p),(q,q) \implies p$ is an isolated symbol (cannot be repeated)

$n = 3,\ (p,q,q),(q,p,q),(q,q,p)$ and (q,q,q) (or (p,q,p))

2. *2-d* sequences:

$\forall n\ \exists k$ s.t. all words of length n contain k or $k+1$ of symbols q

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sequence is *reducible* if one symbol is isolated

reduction = removal of one symbol after each isolated symbol

$\dots p\,q\,q\,p\,q\,q\,q\,p\,q\,q\,q\,p \dots \longrightarrow \dots p\;\; q\,p\;\; q\,q\,p\;\; q\,q\,p \dots$
$\longrightarrow \dots p\;\; p\;\; q\,p\;\; q\,p \dots$

$\dots$	p	q	q	p	q	q	q	p	q	q	q	p	$\dots$
$\dots$	p		q	p		q	q	p		q	q	p	$\dots$
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3. *characteristic* sequences: = *infinitely reducible*

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...	p	q	q	p	q	q	q	p	q	q	q	p	...
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$$
\begin{array}{cccccccccccccc}
\dots & p & q & q & p & q & q & q & p & q & q & q & p & \dots \\
\dots & p &   & q & p &   & q & q & p &   & q & q & p & \dots \\
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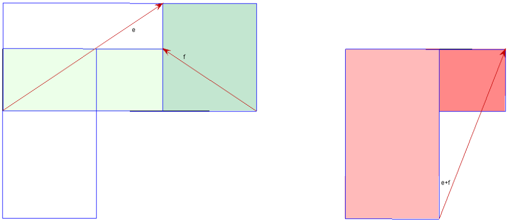

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horizontal projection of all lattice points
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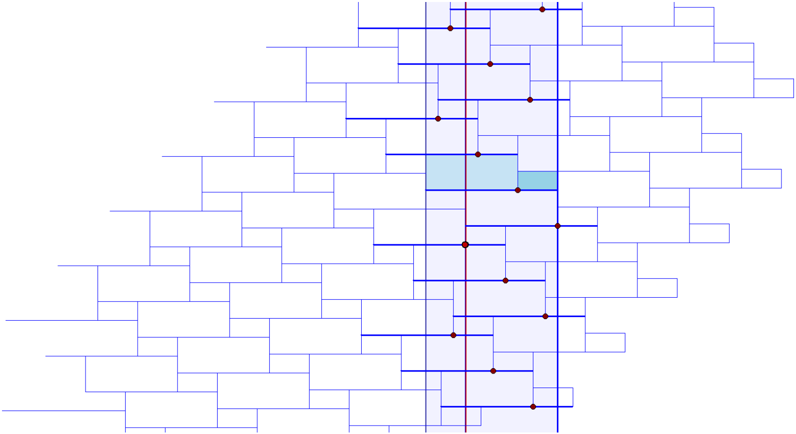

Symmetries of bi-partitions

symmetry = automorphism $A \in Aut(\mathbb{T}^2)$
which takes a bi-partition into a remodelled bi-partition

$\Leftrightarrow$ eigenvectors of A are vertical or horizontal

$\Leftrightarrow$ periodic cutting sequence

$\Longrightarrow$ 1−dimensional *substitution tilings*

partition into $\{K_1, K_2\}$ is *Berg partition* for A if
$A(J^s) \subset J^s$ and $J^u \subset A(J^u)$

Berg partition is a special case of a Markov partition, a notion introduced and developed by Adler-Weiss, Sinai, Bowen

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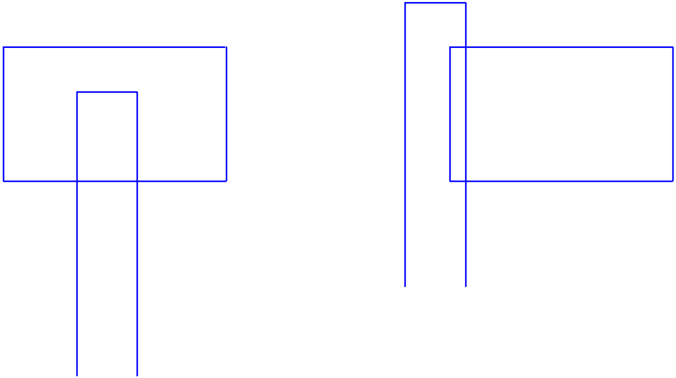
Markov property excludes the “wrong” intersections of K_i and $A(K_j)$



Berg partition comes with transition matrix $M \in Gl(2, \mathbb{Z})$

$M = A^T = \begin{bmatrix} k & m \\ l & n \end{bmatrix}$,

Adler, Manning

to obtain a Berg partition translate 0
so that both spines J^s and J^u contain fixed points
number of fixed points = $|tr\ A - 1|$

Theorem (Siemaszko- W. (2011))

The number of nonequivalent Berg partitions with a given transition matrix $\begin{bmatrix} k & m \\ l & n \end{bmatrix}$ *is equal to* $\left[\frac{k+l+m+n}{2}\right]$.

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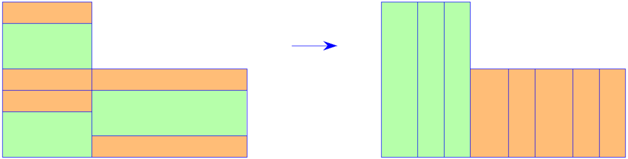
Berg partition with transition matrix $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and substitution

$p \longrightarrow [p\,q]\,p \quad , \quad q \longrightarrow [q\,p]\,p\,q\,p$



New proof based on 1-dim substitution tilings

For a Berg partition the automprphism A^{-1} takes the 1-dimensional tilings into themselves via constant substitutions

$p \longrightarrow$ word with k symbols p and m symbols q

$q \longrightarrow$ word with l symbols p and n symbols q

Substitutions in Berg partitions have the "3 palindromes" property

wol utyl i ma mily tulow
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"3 palindromes" property: denote $k + m = t, l + n = r$

1. consider a palindromic word $[W]$ with $t + r - 2$ symbols

2. such that its first $t - 2$ symbols form a palindrome

3. and its first $r - 2$ symbols also form a palindrome

introduce [lock] = $[p, q]$ or $[q, p]$

consider a subword of the word [W] [lock] [W]
with $t + r$ symbols containing the lock

it has the structure

[last s symbols of W] [lock] [first $t + r - 2 - s$ symbols of W]

split it into consecutive words with t and r symbols

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Example of "3 palindromes" property:

$$\begin{bmatrix} k & m \\ l & n \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$$

3 palindromes of lengths $21, 11$ and 8

there is only one such palindrome!

q q p q q p q q p q q q p q q p q q p q q
q q p q q p q q p q q
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p ⟶ [p q] q q p q q |p| q q p q q
q ⟶ [q p] q q p q | q p q q

Example of “3 palindromes” property:

$$\begin{bmatrix} k & m \\ l & n \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$$

3 palindromes of lengths $21, 11$ and 8

there is only one such palindrome!

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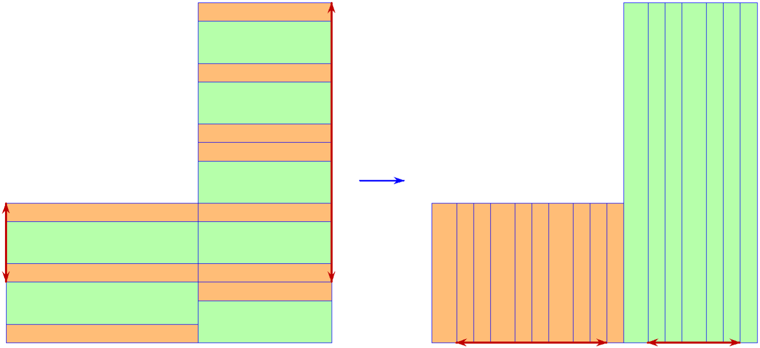
q $\longrightarrow$ [q p] q q p q | q p q q

Berg partition with transition matrix $\begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$ and substitution

p ⟶ [p q] q q p q q |p| q q p q q

q ⟶ [q p] q q p q | q p q q



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$\forall$ relatively prime t, r $\exists!$ palindromic word of length $t + r - 2$ such that its first $t - 2$ symbols and the first $r - 2$ symbols are palindromes.

solution submitted by Allan Pedersen '88, and others
de Luca - Mignosi '94 called it R property in their study of Sturmian words (42 citations in math.sci.net)
Wen-Wen '94 proved that subsititutions preserving Sturmian words are related to automorphisms of the free group with 2 generators F_2
Seebold '98 proved there are exactly k+m+l+n-1 different substitutions preserving a Sturmian word.
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