On the complete integrability of the Ostrovsky-Vakhnenko equation

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Motivation

In 1998 V.O. Vakhnenko investigated high-frequency perturbations in a relaxing barothropic medium. He discovered that this phenomenon is described by a new nonlinear evolution equation. Later it was proved that this equation is equivalent to the reduced Ostrovsky equation, which describes long internal waves in a rotating ocean. The nonlinear integro-differential Ostrovsky-Vakhnenko equation

$$u_t = -uu_x - D_x^{-1}u \qquad (1)$$

on the real axis $\mathbb{R}$ for a smooth function $u \in C^{(\infty)}(\mathbb{R}^2;\mathbb{R})$, where D_x^{-1} is the inverse-differential operator to $D_x := \partial/\partial x$, can be derived as a special case of the Whitham type equation

$$u_t = -uu_x + \int_{\mathbb{R}} K(x,y)u_y dy. \qquad (2)$$

Motivation

Recently J.C. Brunelli and S. Sakovich demonstrated that Ostrovsky-Vakhnenko equation is a suitable reduction of the well known Camassa-Holm equation that made it possible to construct the corresponding compatible Poisson structures for (1), but in a complicated enough non-polynomial form.

In the present work we will reanalyze the integrability of equation (1) both from the gradient-holonomic, symplectic and formal differential-algebraic points of view. As a result, we will re-derive the Lax type representation for the Ostrovsky-Vakhnenko equation (1), construct the related simple enough compatible polynomial Poisson structures and an infinite hierarchy of conservation laws.

Gradient-holonomic integrability analysis

Consider the nonlinear Ostrovsky-Vakhnenko equation (1) as a suitable nonlinear dynamical system

$$du/dt = -uu_x - D_x^{-1}u := K[u] \tag{3}$$

on the smooth 2π-periodic functional manifold

$$M := \{u \in C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}) : \int_0^{2\pi} udx = 0\}, \tag{4}$$

where $K : M \to T(M)$ is the corresponding well-defined smooth vector field on M.

We state that the dynamical system (3) on manifold M possesses an infinite hierarchy of conservation laws, that can signify as a necessary condition for its integrability. For this we need to construct a solution to the Lax gradient equation

$$\varphi_t + K^{',*}\varphi = 0, \tag{5}$$

in the special asymptotic form

$$\varphi = \exp[-\lambda t + D_x^{-1}\sigma(x;\lambda)], \tag{6}$$

where, by definition, a linear operator $K^{',*} : T^*(M) \to T^*(M)$ is, adjoint with respect to the standard convolution $(\cdot,\cdot)$ on $T^*(M) \times T(M)$, the Frechet-derivative of a nonlinear mapping $K : M \to T(M)$:

$$K^{',*} = uD_x + D_x^{-1} \tag{7}$$

and, respectively,

$$\sigma(x;\lambda) \simeq \sum_{j\in\mathbb{Z}_+} \sigma_j[u]\lambda^{-j}, \tag{8}$$

as $|\lambda| \to \infty$ with some "local" functionals $\sigma_j : M \to C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ on M for all $j \in \mathbb{Z}_+$.

By substituting (6) into (5) one easily obtains the following recurrent sequence of functional relationships:

$$\sigma_{j,t} + \sum_{k \leq j} \sigma_{j-k}(u\sigma_k + D_x^{-1}\sigma_{k,t}) - \sigma_{j+1} + (u\sigma_j)_x + \delta_{j,0} = 0 \qquad (9)$$

for all $j+1 \in \mathbb{Z}_+$ modulo the equation (3). By means of standard calculations one obtains that this recurrent sequence is solvable and

$$\begin{aligned} \sigma_0[u] &= 0, \sigma_1[u] = 1, \sigma_2[u] = u_x, \\ \sigma_3[u] &= 0, \sigma_4[u] = u_t + 2uu_x, \\ \sigma_5[u] &= 3/2(u^2)_{xt} + u_{tt} + 2/3(u^3)_{xx} - u_x D_x^{-1} u \end{aligned} \qquad (10)$$

and so on. It is easy check that all of functionals

$$\gamma_j := \int_0^{2\pi} \sigma_j[u]dx \qquad (11)$$

are on the manifold M conservation laws, that is $d\gamma_j/dt = 0$ for $j \in \mathbb{Z}_+$ with respect to the dynamical system (3).

So we can suggest that the dynamical system (3) on the functional manifold M is an integrable Hamiltonian system.
We show that this dynamical system is a Hamiltonian flow

$$du/dt = -\vartheta \; grad \; H[u] \tag{12}$$

with respect to some Poisson structure $\vartheta : T^*(M) \to T(M)$ and a Hamiltonian function $H \in \mathcal{D}(M)$. Consider the conservation law in the scalar "momentum" form:

$$-1/2\gamma_5 = \frac{1}{2}\int_0^{2\pi} u_x D_x^{-1} u dx = (1/2 D_x^{-1} u, u_x) := (\psi, u_x) \tag{13}$$

with the co-vector $\psi := 1/2 D_x^{-1} u \in T^*(M)$ and calculate the corresponding co-Poissonian structure

$$\vartheta^{-1} := \psi' - \psi'^{,*} = D_x^{-1}, \tag{14}$$

or the Poissonian structure

$$\vartheta = D_x. \tag{15}$$

The obtained operator $\vartheta = D_x : T^*(M) \to T(M)$ is really Poissonian for (3) since the following determining symplectic condition

$$\psi_t + K^{',*}\psi = grad\ \mathcal{L} \tag{16}$$

holds for the Lagrangian function

$$\mathcal{L} = \frac{1}{12}\int_0^{2\pi} u^3 dx. \tag{17}$$

As a result of (16) one obtains easily that

$$du/dt = -\vartheta grad\ H[u], \tag{18}$$

where the Hamiltonian function

$$H = (\psi, K) - \mathcal{L} = \frac{1}{2}\int_0^{2\pi}[u^3/3 - (D_x^{-1}u)^2/2]dx \tag{19}$$

is an additional conservation law of the dynamical system (3). Thus, one can formulate the following proposition.

Proposition

The Ostrovsky-Vakhnenko dynamical system (3) possesses an infinite hierarchy of nonlocal, in general, conservation laws (11) and is a Hamiltonian flow (18) on the manifold M with respect to the Poissonian structure (15).

It is useful to remark here that the existence of an infinite ordered by λ-powers hierarchy of conservations laws (11) is a typical property of the Lax type integrable Hamiltonian systems, which are simultaneously bi-Hamiltonian flows with respect to corresponding two compatible Poissonian structures.

As is well known, the second Poissonian structure $\eta : T^*(M) \rightarrow T(M)$ on the manifold M for (3), if it exists, can be calculated as

$$\eta^{-1} := \tilde{\psi}' - \tilde{\psi}'^{,*}, \tag{20}$$

where a-covector $\tilde{\psi} \in T^*(M)$ is a second solution to the determining equation (16):

$$\tilde{\psi}_t + K^{',*}\tilde{\psi} = grad\ \tilde{\mathcal{L}} \tag{21}$$

for some Lagrangian functional $\tilde{\mathcal{L}} \in \mathcal{D}(M)$.
We apply the direct differential-algebraic approach to dynamical system (3) and reveal its Lax type representation both in the differential scalar and in canonical matrix Zakharov-Shabat forms. Next, we construct the naturally related compatible polynomial Poissonian structures for Ostrovsky -Vakhnenko dynamical system (3) and generate an infinite hierarchy of commuting to each other nonlocal conservation laws.

We found the differential Lax type relationships

$$D_x^3 f = -\mu \bar{u} f, \; D_x^3 \bar{f} = \mu \bar{u} \bar{f}, \tag{22}$$

and

$$D_t f = \mu^{-1} D_x^2 f + u_x f, \; D_t \bar{f} = -\mu^{-1} D_x^2 \bar{f} - 2 u_x \bar{f}, \tag{23}$$

where $\bar{u} := u_{xx} + 1/3, \mu \in \mathbb{C} \backslash \{0\}$ is an arbitrary complex parameter, hold. Moreover, they exactly coincide with those found before in [4].

The obtained above differential relationships (22) and (23) can be equivalently rewritten in the following matrix Zakharov-Shabat type form:

$$D_t h = \hat{q}[u;\mu]h, \ \ D_x h = \hat{l}[u;\mu]h, \tag{24}$$

where matrices

$$\hat{q}[u;\mu] := \begin{pmatrix} u_x & 0 & 1/\mu \\ -1/3 & 0 & 0 \\ 0 & -1/3 & -u_x \end{pmatrix}, \quad \hat{l}[u;\mu] := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu\bar{u} & 0 & 0 \end{pmatrix} \tag{25}$$

and $h := (f, D_x f, D_x^2 f)^\intercal \in \mathcal{K}\{u\}^3$.

Based further on the obtained differential relationships (22) and (23), one obtains that the following important relationship

$$-\vartheta\varphi = D_x^2 D_t \varphi = 3\mu^2 \eta \varphi, \tag{26}$$

holds, where the polynomial integro-differential operator

$$\eta := \partial^{-1}\bar{u}\partial^{-3}\bar{u}\partial^{-1} + 4\partial^{-2}\bar{u}\partial^{-1}\bar{u}\partial^{-2} + 2(\partial^{-2}\bar{u}\partial^{-2}\bar{u}\partial^{-1} + \partial^{-1}\bar{u}\partial^{-2}\bar{u}\partial^{-2}) \tag{27}$$

is skewsymmetric on the functional manifold M and presents the second compatible Poisson structure for the Ostrovsky-Vakhnenko dynamical system (3).

Based now on the recurrent relationships following from substitution of the asymptotic expansion

$$\varphi \simeq \sum_{j \in \mathbb{Z}_+} \varphi_j \xi^{-j}, \ \xi := -1/(3\mu^2), \tag{28}$$

into (26), one can determine a new infinite hierarchy of conservations laws for dynamical system (3):

$$\tilde{\gamma}_j := \int_0^1 ds(\varphi_j[us], u), \tag{29}$$

for $j \in \mathbb{Z}_+$, where

$$\varphi_j = \Lambda^j \varphi_0, \ \vartheta\varphi_0 = 0, \tag{30}$$

and the recursion operator $\Lambda := \vartheta^{-1}\eta : T^*(M) \to T^*(M)$ satisfies the standard Lax type representation:

$$\Lambda_t = [\Lambda, K^{'*}]. \tag{31}$$

Proposition

The Ostrovsky-Vakhnenko dynamical system (3) allows the standard differential Lax type representation (22), (23) and defines on the functional manifold M an integrable bi-Hamiltonian flow with compatible Poisson structures (15) and (27). In particular, this dynamical system possesses an infinite hierarchy of nonlocal conservation laws (29), defined by the gradient elements (30).

It is useful to remark here that the existence of an infinite λ-powers ordered hierarchy of conservations laws (11) is a typical property of the Lax type integrable Hamiltonian systems, which are simultaneously bi-Hamiltonian flows with respect to corresponding compatible Poissonian structures.
It is interesting to observe that our second polynomial Poisson structure (27) differs from that obtained recently in [3], which contains the rational power factors.

It is easy to construct making use of the differential expressions (22) and (23) a slightly different from (24) matrix Lax type representation of the Zakharov-Shabat form for the dynamical system (1).
Really, if to define the "spectral" parameter $\mu := 1/(9\lambda) \in \mathbb{C}\backslash\{0\}$ and new basis elements of the invariant differential ideal:

$$g_1 := -3D_x f,\ g_2 := f,\ g_3 := 9\lambda D_x^2 f + u_x f, \tag{32}$$

The relationships (22) and (23) can be rewritten as follows:

$$D_t g = q[u;\lambda]g,\ D_x g = l[u;\lambda]g, \tag{33}$$

where matrices

$$q[u;\lambda] := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -u & 0 \end{pmatrix}, l[u;\lambda] := \begin{pmatrix} 0 & u_x/(3\lambda) & -1/(3\lambda) \\ -1/3 & 0 & 0 \\ -u_x/3 & -1/3 & 0 \end{pmatrix} \tag{34}$$

coincide with those of [4, 3] and satisfy the following Zakharov-Shabat type compatibility condition:

$$D_t l = [q, l] + D_x q - l\ D_x u. \tag{35}$$

Remark As it was already mentioned above, the Lax type representation (34) of the Ostrovsky-Vakhnenko dynamical system (1) was obtained in [4] by means of a suitable limiting reduction of the Degasperis-Processi equation

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{36}$$

For convenience, let us rewrite the latter in the following form:

$$D_t z = -3zD_x u, \quad z = u - D_x^2 u, \tag{37}$$

where differentiations $D_t := \partial/\partial t + u\partial/\partial x$ and $D_x := \partial/\partial x$ satisfy the Lie-algebraic relationship $[D_x, D_t] = u_x D_x$. It appears to be very impressive that equation (36) is itself a special reduction of a new Lax type integrable Riemann type hydrodynamic system, proposed and studied (at $s = 2$) recently in [6]:

$$D_t^{N-1} u = \bar{z}_x^s \ , \quad D_t \bar{z} = 0, \tag{38}$$

where $s,\ N \in \mathbb{N}$ are arbitrary natural numbers.

Really, having put, by definition, $z := \bar{z}_x^s$ and $s = 3$, from (38) one easily obtains the following dynamical system:

$$\begin{aligned} D_t^{N-1} u &= z, \\ D_t z &= -3zD_x u, \end{aligned} \tag{39}$$

coinciding with the Degasperis-Processi equation (37) if to make the identification $z = u - D_x^2 u$. As a result, we have stated that a function $u \in C^\infty(\mathbb{R}^2; \mathbb{R})$, satisfying for an arbitrary $N \in \mathbb{N}$ the generalized Riemann type hydrodynamical equation $D_t^{N-1} u = u - D_x^2 u$,simultaneously solves the Degasperis-Processi equation (36). In particular, having put $N = 2$, we obtain that solutions to the Burgers type equation $D_t u = u - D_x^2 u$are solving also the Degasperis-Processi equation (36). It means, in particular, that the reduction procedure of the work [4] can be also applied to the Lax type integrable Riemann type hydrodynamic system (38), giving rise to a related Lax type representation for the Ostrovsky-Vakhnenko dynamical system (1).

Conclusion

We have showed that the Ostrovsky-Vakhnenko dynamical system is naturally embedded into the general Lax type integrability scheme [5], whose main ingredients such as the corresponding compatible Poissonian structures and Lax type representation can be effectively enough retrieved by means of direct modern integrability tools, such as the differential-geometric, differential-algebraic and symplectic gradient holonomic approaches. We have also demonstrated the relationship of the Ostrovsky-Vakhnenko equation (1) with a generalize Riemann type hydrodynamic system, studied recently in [6] and its reduction.

References

- Ostrovsky L.A. Nonlinear Internal Waves in a Rotating Ocean. Okeanologia. 18, 181 (1978)
- Vakhnenko V.A. Solitons in a Nonlinear Model Medium. J. Phys. 25A, 4181 (1992)
- Brunelli J.C., Sakovich S. Hamiltonian Structures for the Ostrovsky-Vakhnenko Equation. arXiv:1202.5129v1 [nlin.SI] 23 Feb 2012
- Degasperis A., Holm D.D. and Hone A.N.W. A New Integrable Equation with Peakon Solutions. Theor. Math. Phys. 133, 1463 (2002).
- Prykarpatsky A. and Mykytyuk I. Algebraic Integrability of nonlinear dynamical systems on manifolds: classical and quantum aspects. Kluwer Academic Publishers, the Netherlands, 1998
- Blackmore D., Prykarpatrsky Y.A., Artemovych O.D., Prykarpatsky A.K. On the complete integrability of a one generalized Riemann type