Old and New Results on Biharmonic Submanifolds in Spheres

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Definitions

Harmonic maps

critical points $\varphi : (M, g) \to (N, h)$ of

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

[ES64, EL83]

Euler-Lagrange Equation of E

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi = 0$$

Biharmonic maps

critical points $\varphi : (M, g) \to (N, h)$ of

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

[J86]

Euler-Lagrange Equation of E_2

$$\tau_2(\varphi) = -\Delta\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0.$$

Biharmonic submanifolds $\imath : M^m \to \mathbb{R}^n$

$$\tau_2(\imath) = 0 \Leftrightarrow \Delta H = 0 \Leftrightarrow \Delta^2 \imath = 0$$

Chen's Conjecture

Any biharmonic submanifold $\imath : M^m \to \mathbb{R}^n$ in the Euclidean space is minimal.

Chen's Conjecture holds for:

- curves in $\mathbb{R}^n$ [D92]
- hypersurfaces with $k \leq 2$ in $\mathbb{R}^n$ [D92]
- hypersurfaces in $\mathbb{R}^4$ [HV95]
- submanifolds of finite type in $\mathbb{R}^n$ [D92]
- spherical submanifolds in $\mathbb{R}^n$ [C91]

Generalized Chen's conjecture

Any biharmonic submanifold $\imath : M^m \to N^n$, $\operatorname{Riem}^N \leq 0$, is minimal.

- holds when M is complete and $\imath$ is of finite bienergy [NU11]
- there are counterexamples when Riem^N is non-constant [OT10]
- still open when Riem^N is constant

Biharmonic submanifolds $\imath : M^m \to \mathbb{S}^n$ - characterization results

$$\tau_2(\imath) = 0$$

$\Updownarrow$

$$\Delta H = mH$$

[O03]

Tangent and normal components of $\tau_2(\imath)$

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) - mH = 0 \\ 4 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + m \operatorname{grad}(|H|^2) = 0 \end{cases}$$

[BO12]

PMC biharmonic submanifolds $M^m \subset \mathbb{S}^n$

$$\begin{cases} |A_H|^2 = m|H|^2 \\ \langle A_H, A_\eta \rangle = 0, \quad \forall \eta \in C(NM), \eta \perp H \end{cases}$$

Biharmonic hypersurfaces $M^m \subset \mathbb{S}^{m+1}$

$$\begin{cases} \Delta f = (m - |A|^2) f \\ A(\operatorname{grad} f) = -\dfrac{m}{2} f \operatorname{grad} f \end{cases}$$

CMC biharmonic hypersurfaces $M^m \subset \mathbb{S}^{m+1}$

$$|A|^2 = m$$

Biharmonic submanifolds $\imath : M^m \to \mathbb{S}^n$ - main examples and classification results

Biharmonic hypersurfaces $M^m \subset \mathbb{S}^{m+1}$

isoparametric

CMC with $\operatorname{Riem}^M \geq 0$

k = number of distinct principal curvatures

[IIU08, BMO12]

$k = 1$ $\quad$ $k \leq 2$ $\quad$ $k = 3$

$M^3 \subset \mathbb{S}^4$ — [BMO10] → CMC biharmonic submanifolds

compact

compact CMC — [BMO10] → nonexistence

[BMO08] [CMO01]

CMC biharmonic submanifolds $M^m \subset \mathbb{S}^n$

[O03] $|H| = 1$ — [BMO12] → M of 1-type, center of mass of norm $\frac{1}{\sqrt{2}}$ and with eigenvalue $2m$

[O03] $|H| \in (0,1)$ — [BMO12] → M of 2-type, mass-symmetric and with eigenvalues $m(1 \pm |H|)$

[O02]

$M^2 \subset \mathbb{S}^4$ — [BO09] → PMC biharmonic submanifolds

$M^m \subset \mathbb{S}^{m+1}$

PMC biharmonic submanifolds $M^m \subset \mathbb{S}^n$

$M^2 \subset \mathbb{S}^n$ [BMO08]

$|H| \neq 1$

[BO12] $|H| = \frac{m-2}{m}$ $\quad$ [BO12] $|H| \in (0, \frac{m-2}{m})$

[BO12]

$$M \overset{\text{locally}}{=} M_1^{m-1} \times \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) \xrightarrow{\text{minimal}} \mathbb{S}^{n-2}(\tfrac{1}{\sqrt{2}}) \times \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) \longrightarrow \mathbb{S}^n \quad \text{(biharmonic)}$$

$\nabla^\perp B = 0$ → $\nabla A_H = 0$ — [BO12]

Biharmonic hypersphere [CMO01]

$$M \overset{\text{open}}{\subseteq} \mathbb{S}^m(\tfrac{1}{\sqrt{2}}) \xrightarrow{\text{biharmonic}} \mathbb{S}^{m+1}$$

Biharmonic product of spheres [J86]

$$M \overset{\text{open}}{\subseteq} \mathbb{S}^{m_1}(\tfrac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\tfrac{1}{\sqrt{2}}) \xrightarrow{\text{biharmonic}} \mathbb{S}^{m+1}$$

$$m_1 + m_2 = m, \ m_1 \neq m_2$$

Composition property [CMO02]

$$M^m \xrightarrow{\text{minimal}} \mathbb{S}^{n-1}(\tfrac{1}{\sqrt{2}}) \xrightarrow{\text{biharmonic}} \mathbb{S}^n \quad \text{(biharmonic)}$$

Product composition property [CMO02]

$$M_1^{m_1} \times M_2^{m_2} \xrightarrow{\text{minimal}} \mathbb{S}^{n_1}(\tfrac{1}{\sqrt{2}}) \times \mathbb{S}^{n_2}(\tfrac{1}{\sqrt{2}}) \longrightarrow \mathbb{S}^n \quad \text{(biharmonic)}$$

$$n_1 + n_2 = n - 1, \ m_1 \neq m_2$$

classics $\quad$ by composition

Main examples

curves in $\mathbb{S}^3$

Circles [CMO01]

$$\mathbb{S}^1(\tfrac{1}{\sqrt{2}}) \xrightarrow{\text{biharmonic}} \mathbb{S}^3$$

Geo of the Clifford Torus [CMO01]

$$\gamma \xrightarrow{\text{geo}} \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) \times \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) \longrightarrow \mathbb{S}^3 \quad \text{(biharmonic)}$$

γ geodesic of slope $\neq \pm 1, 0$ and ∞

[CMO02] $\kappa = 1$ $\quad$ [CMO02] $\kappa \in (0, 1)$

κ = curvature

Biharmonic curves in $\mathbb{S}^n$

Open problems

Conjecture 1. The only proper-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ are the open parts of hyperspheres $\mathbb{S}^m(1/\sqrt{2})$ or of the standard products of spheres $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Statements equivalent to Conjecture 1.

- A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ has at most two principal curvatures everywhere.
- A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ is parallel.
- A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ is CMC and has non-negative sectional curvature.
- A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ is isoparametric.

Conjecture 2. The proper-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ are CMC.

Problem. Find a PMC proper-biharmonic submanifold M^m in $\mathbb{S}^n$ such that A_H is not parallel.

Notations and conventions

This poster illustrates the known results on the classification of biharmonic submanifolds in $\mathbb{S}^n$.

- biharmonic submanifold = biharmonic isometric immersion
- proper-biharmonic = biharmonic non-harmonic
- $\Delta^2 = \Delta \circ \Delta$
- $\Delta = -\operatorname{trace}_g (\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla)$ is the rough Laplacian.
- $R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature on N.
- B is the second fundamental form of M in $\mathbb{S}^n$.
- H is the mean curvature vector field of M^m in $\mathbb{S}^n$.
- A_H is the Weingarten operator of M^m associated to H.
- A is the shape operator of M^m in $\mathbb{S}^{m+1}$ and $f = \operatorname{trace} A/m$.
- k is the number of distinct principal curvatures of a hypersurface.

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