# Some mathematical aspects of water waves

Adrian Constantin adrian.constantin@kcl.ac.uk
*King's College London and University of Vienna*

## Abstract

Water waves have fascinated scientists and laypersons alike since times immemorial and their understanding extends beyond intellectual curiosity. This research area is a rich source of problems where progress is contingent upon a fruitful interplay between rigorous mathematical analysis, numerical simulation, experimental evidence, and physical intuition. Moreover, in water-wave phenomena nonlinear approaches often describe more accurately the real nature of the ongoing processes instead of linear paradigms that usually capture only small perturbations of simple states.

The aim of this talk is to survey recent advances on two fundamental aspects of water-wave theory:

1. While watching the sea it is often possible to trace a wave as it propagates on the waters surface. Contrary to a possible first impression, what one observes traveling across the sea is not the water but a pattern (pulse of energy), as enunciated intuitively in the fifteenth century by Leonardo da Vinci in the following form: "...it often happens that the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in place." A basic question in water waves concerns the flow beneath a surface wave. Stokes waves are the most regular wave patterns propagating at constant speed at the surface of water in irrotational flow over a flat bed. It is widely believed (see, for example, any classical textbook on water waves) that particles in the water beneath a Stokes wave execute a closed-path motion as the surface wave passes over: individual particles of water do not travel along with the wave, but instead they move in closed, circular or elliptical, orbits. Support for this conclusion is apparently given by experimental evidence: photographs of small buoyant particles in laboratory wave tanks where almost closed elliptical paths are recognizable. The classical approach towards explaining this aspect of water waves consists in analyzing the particle motion after linearizing the nonlinear governing equations for water waves. But even after linearizing the governing equations and obtaining explicit formulas for the free surface and for the fluid velocity field, the system describing particle motion turns out to be again nonlinear. Thus one linearizes again and the closed paths emerge. However, it turns out that no particle trajectory is closed: over a wave period, each particle that does not lie on the flat bed performs

a backward/forward and upward/downward movement, with the path an elliptical-like loop, not closed but with a forward drift (albeit mostly small and thus often hard to detect experimentally). On the flat bed this path degenerates into a back-and-forth horizontal motion. This fine feature is lost in the process of linearization but can be established within the framework of the nonlinear governing equations. The proof relies on an interplay of methods from harmonic function theory, dynamical systems and elliptic partial differential equations.

2. Of all the magnificent scenes presented by water waves, breaking is surely among the most impressive. Although the mathematical description of the processes of breaking could hardly be regarded as satisfactory, some theoretical investigations offer insight into this fundamental aspect of water waves. The complexity of the governing equations prevent an investigation of breaking waves within this framework. One is thus led to the derivation of approximations using simplifying assumptions such as "small amplitude", "shallow water", "unidirectionality" within certain regimes, rendering mathematical models of various degrees of sophistication amenable to a more detailed analysis. The obtained simplified model equations are linear to the lowest order of approximation but higher-order approximations incorporate nonlinear effects. These models usually have features which make them suitable to explain certain observations. Linear theory does not capture the breaking wave phenomenon, nor does the weakly nonlinear theory of shallow water waves of small amplitude (the latter being the setting in which an intensive research activity over the last 30 years led to remarkable success in the understanding of integrable equations with soliton solutions). Promising recent insight in this direction was provided by studying breaking waves based on the Camassa-Holm equation which arises as an approximation to the governing equations for water waves in the shallow water regime of waves of moderate amplitude. We will discuss this aspect.

Both themes illustrate that an appreciation of mathematical rigor and elegance, combined with the power of meaningful abstraction, often leads to breakthroughs in physical insight, while mathematics draws considerable inspiration and stimulation from physics.

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