Ring of flag vectors

Nickolai Erokhovets erokh@higeom.math.msu.su
Lomonosov Moscow State University and Delone Laboratory of Discrete and Computational Geometry, Jaroslavl State University, Russia
Coauthors: Victor Buchstaber

Abstract

The famous g-theorem of R.P.Stanley, L.Billera and C.Lee gives the necessary and sufficient condition for the set of numbers $(f_0, \dots, f_{n-1}) \in \mathbb{Z}^n$ to be the face vector of a simple convex polytope. The analogous problem for arbitrary polytopes involves flag numbers and is opened even in dimension four. We present the new effective approach based on the ring of convex polytopes introduced by Victor Buchstaber. Let $\mathfrak{P}$ be the free abelian group generated by all combinatorial convex polytopes. It is a commutative ring with the multiplication given by the direct product of polytopes. The ring $\mathfrak{F} = \mathfrak{P}/\sim$, where $P \sim Q$ if P and Q have equal flag numbers, is called the ring of flag vectors. The face operators d_k that send a polytope P to the sum of its faces of codimension k allows one to apply the technique of Hopf algebras and quasisymmetric functions to the problem of flag numbers. We show that $\mathfrak{F} \otimes \mathbb{Q}$ is a ring of polynomials, find the embedding $\mathfrak{F} \subset \mathcal{Q}\mathrm{sym}[\alpha]$, find the basis in the free abelian group $\mathfrak{F}$ using the operators of pyramid and bipyramid that gives the cd-index of J.Fine, and find the pure algebraic construction of g-polynomial of deformation of multiplication in the graded ring that gives the well-known toric g-polynomial.

AMS Classification: Primary 52B05; Secondary 16T05.