Combinatorial realisation of cycles and small covers

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Abstract

The following problem on realisation of cycles was posed by Steenrod in 1940s. Given a homology class $x \in H_n(X, \mathbb{Z})$ of a topological space X, does there exist an oriented closed smooth manifold M^n and a continuous mapping $f : M^n \to X$ such that $f_*[M^n] = x$? If the answer is "yes", x is said to be realisable. In 1954, Thom found a non-realisable 7-dimensional class and proved that for every n, there is a positive integer $k(n)$ such that the class $k(n)x$ is always realisable. The proof was by methods of algebraic topology and gave no information on the topology of M^n. We give a purely combinatorial construction of a manifold that realises a multiple of a given homology class. For every n, this allows us to find a manifold M^n that has the following universality property:

($*$) For any X and any $x \in H_n(X, \mathbb{Z})$, some multiple of x can be realised by an image of some non-ramified finite-sheeted covering of M^n.

This manifold M^n is a so-called small cover of the permutahedron, i.e., a manifold glued in a special way out of 2^n permutahedra. (The permutahedron is a special convex polytope with $n!$ vertices.) Further, among small covers over other simple polytopes, we find a broad class of examples of manifolds that have property ($*$). In particular, in dimension 4, we find a hyperbolic manifold with property ($*$), thus proving a conjecture of Kotschick and Löh claiming that a multiple of any homology class can be realised by an image of a hyperbolic manifold.

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