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On the discrepancy of some generalized Kakutani's sequences of partitions

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1. Introduction

This poster presents some recent results [2] concerning the discrepancy of a class of generalized Kakutani's sequences of partitions of the unit interval, namely sequences constructed by using the technique of successive $\rho-$refinements. This procedure has been recently introduced by Volčič in [6] and it is an extension of the Kakutani splitting procedure proposed in [4].

Uniform distribution of sequences of partitions

Let (π_n) be a sequence of partitions of $[0,1]$ represented by $\pi_n = \left\{[t_{i-1}^{(n)}, t_i^{(n)}] : 1 \leq i \leq k(n)\right\}$, where $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = 1$. The sequence (π_n) is said to be **uniformly distributed (u.d.)** if for any continuous function f on $[0,1]$ we have

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f\left(t_i^{(n)}\right) = \int_0^1 f(t)\, dt.$$

Equivalently, (π_n) is u.d. if the sequence of **discrepancies**

$$D_n = \sup_{0\leq a<b\leq 1} \left| \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{[a,b[}\left(t_i^{(n)}\right) - (b-a)\right|$$

tends to 0 as $n \to \infty$ ($\chi_{[a,b[}$ denotes the characteristic function of the set $[a,b[$).

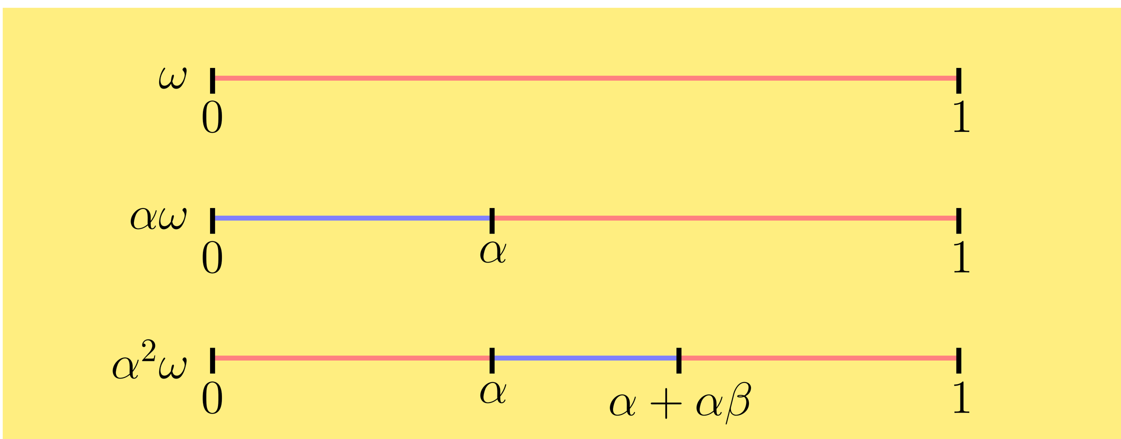
The discrepancy is the classical measure of deviation of a sequence from the ideal uniform distribution. U.d. sequences of points with low discrepancy are essential tools to get faster rates of convergence in quasi-Monte Carlo methods.

2. Kakutani's sequences

Kakutani's $\alpha-$refinement

Let π be any partition of $[0,1]$ and $\alpha \in]0,1[$. Kakutani's α**-refinement** of π (denoted by $\alpha\pi$) is obtained by splitting all the intervals of π having maximal length in two parts proportional to α and $\beta := 1-\alpha$, respectively.

For $\omega := \{[0,1]\}$ the sequence $(\alpha^n\omega)$ is called **Kakutani's sequence of partitions**, [4].



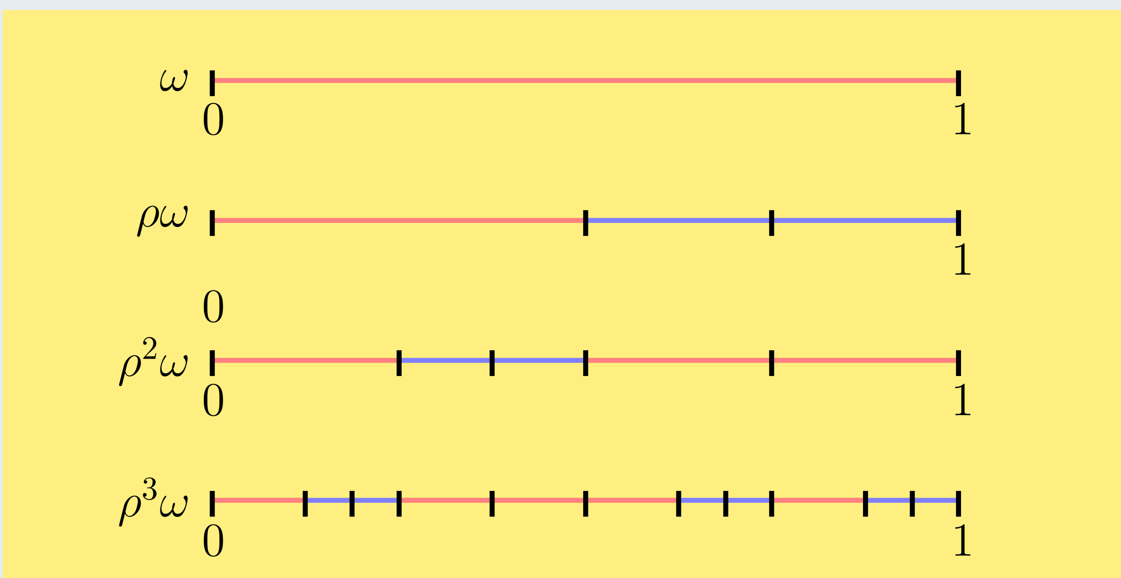
Theorem. (S. Kakutani, 1976)

For any $\alpha \in]0,1[$, Kakutani's sequence $(\alpha^n\omega)$ is u.d..

3. Generalized Kakutani's sequences

ρ-refinement

Let π be any partition of $[0,1]$ and let ρ be a non-trivial partition of $[0,1]$. The ρ**-refinement** of π (denoted by $\rho\pi$) is obtained by splitting all the intervals of π having maximal length into a finite number of parts homothetically to ρ.



If $\rho = \{[0,\alpha[, [\alpha,1]\}$ with $0<\alpha<1$, then $\rho^n\omega = \alpha^n\omega$.

Theorem. (A. Volčič, 2011)

For any non-trivial partition ρ of $[0,1]$, the sequence $(\rho^n\omega)$ is u.d..

4. Problem

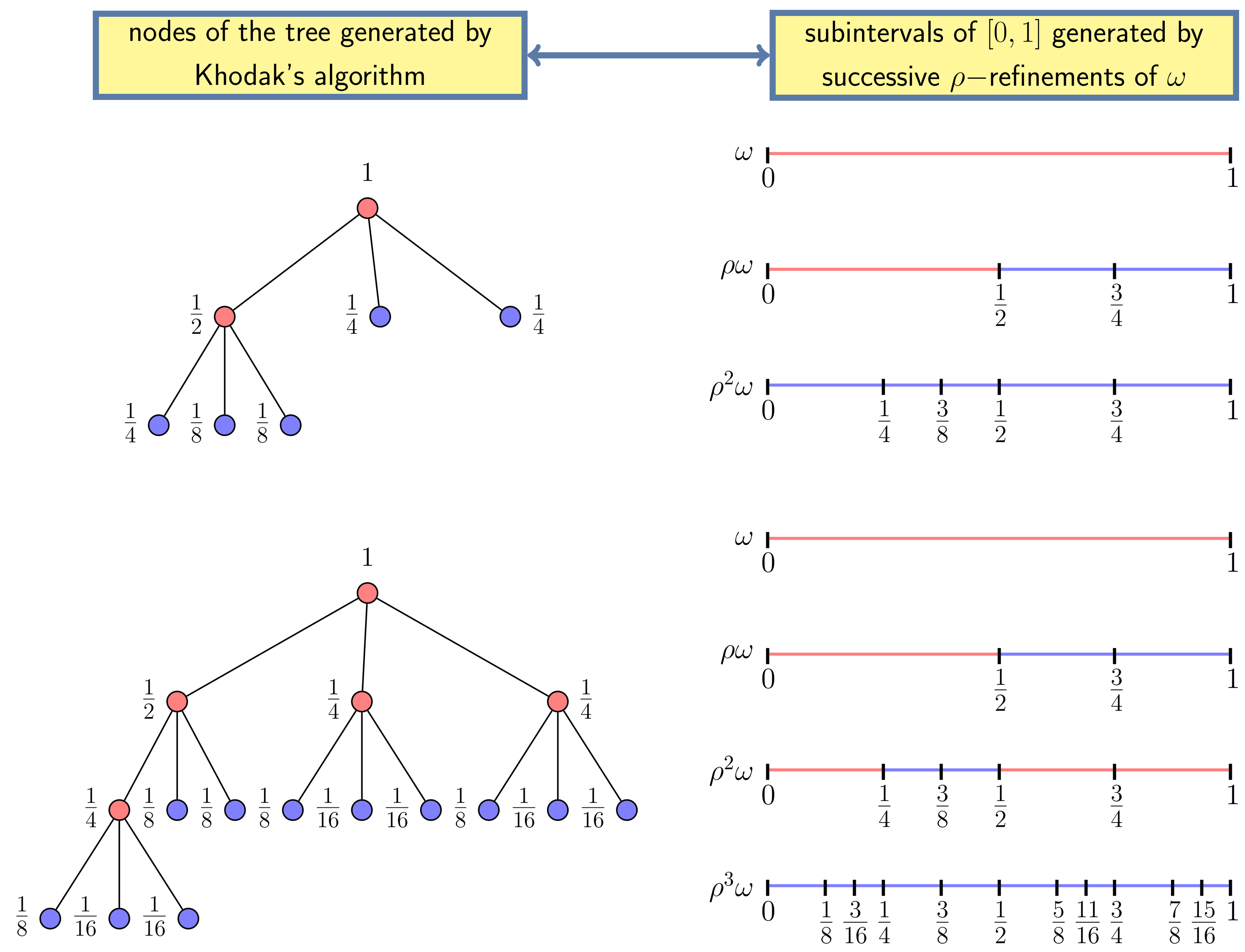
In [2] we provide a quantitative analysis of the distribution behaviour of sequences generated by successive $\rho-$refinements of $[0,1]$. Basically, our main focus is

To estimate the asymptotic behaviour of the discrepancy D_n of a sequence $(\rho^n\omega)$ as $n\to\infty$.

This problem has been first posed in [6] but so far the only known discrepancy bounds for generalized Kakutani's sequences have been obtained in [1] for ρ consisting of L intervals of length α and S intervals of length α^2 with $L\alpha + S\alpha^2 = 1$.

5. Idea: Khodak's algorithm $\leftrightarrow$ $\rho-$refinements

Our new approach is based on a parsing tree (related to Khodak's algorithm [5]), which represents the successive ρ-refinements.



6. Results

Using a refinement of some results in [3] about Khodak's algorithm we get the following estimates.

Theorem. Discrepancy bounds in the rational case, (M. Drmota - M. I., 2012)

Suppose that the lengths of the intervals of a partition ρ of $[0,1]$ are $p_1,\dots,p_m$ and that $\log\left(\frac{1}{p_j}\right)$ for $j=1,\dots,m$ are **rationally related** i.e. commensurable. Then there exist a real number $\eta>0$ and an integer $d\geq 0$ such that the discrepancy D_n of $(\rho^n\omega)$ is bounded by

$$D_n = \begin{cases} \mathcal{O}\left(\frac{(\log k(n))^d}{k(n)^\eta}\right) & \text{if } 0<\eta<1,\\ \mathcal{O}\left(\frac{(\log k(n))^{d+1}}{k(n)}\right) & \text{if } \eta=1,\\ \mathcal{O}\left(\frac{1}{k(n)}\right) & \text{if } \eta>1. \end{cases}$$

Moreover, there exist $\delta>0$ and infinitely many n such that

$$D_n \geq \begin{cases} \delta\,\frac{(\log k(n))^d}{k(n)^\eta} & \text{if } 0<\eta\leq 1,\\ \delta\,\frac{1}{k(n)} & \text{if } \eta>1. \end{cases}$$

Note that the parameters η and d are effectively computable and they only depend on $p_1,\dots,p_m$.

Theorem. Discrepancy bounds in the irrational case, (M. Drmota - M. I., 2012)

Suppose that the lenghts of the intervals of a partition ρ of $[0,1]$ are p and $q=1-p$. If $\frac{\log p}{\log q}\notin\mathbb{Q}$ is badly approximable, then the discrepancy D_n of $(\rho^n\omega)$ is bounded by

$$D_n = \mathcal{O}\left(\left(\frac{\log\log(k(n))}{\log(k(n))}\right)^{\frac{1}{4}}\right), \quad \text{as } n\to\infty.$$

Furthermore, if p,q are algebraic numbers then

$$D_n = \mathcal{O}\left(\left(\frac{\log\log(k(n))}{\log(k(n))}\right)^{\kappa}\right), \quad \text{as } n\to\infty,$$

where κ is an effectively computable positive real constant.

7. Conclusions

Our results provide optimal bounds for the discrepancy of a countable class of $\rho-$refinements (i.e. the rational case) including the ones in [1] and all the classical Kakutani sequences of parameter α such that $\frac{\log\alpha}{\log(1-\alpha)}\in\mathbb{Q}$. On the other hand there are still interesting **open problems**.

- Our upper bounds in the irrational case are closely related to the Diophantine approximation properties of the ratios $\frac{\log p_i}{\log p_j}$ and they are weaker than the estimates in the rational case. The main problem is to understand if this bad behaviour is due to the *method or to the nature of the irrational sequences*.
- We aim to find *explicit algorithms to provide low discrepancy sequences of points associated* to a low discrepancy sequence of partitions constructed by successive $\rho-$refinements.

References

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