Multidimensional periodic and almost-periodic spectral problems

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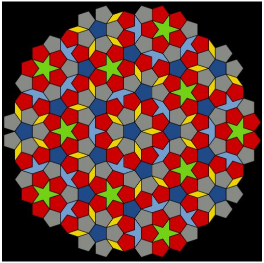

We consider elliptic periodic differential (or even pseudo-differential) operators

$$H = h(x, D),$$

where $x \in \mathbb{R}^d$, $d \geq 2$, and h is either periodic, or almost-periodic in x.

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Another important example is the magnetic Schrödinger operator

$$H = (i\nabla + a)^2 + V$$

with smooth scalar potential $V = V(x)$ and smooth vector potential $a = a(x)$.

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(iii) 'How much' of a spectrum can we have, say in a large interval $[0, \lambda]$?

The standard way to 'count' the spectrum of H is to study the integrated density of states which can be defined by the formula

$$N(\lambda) = N(\lambda; H) := \lim_{L\to\infty} \frac{N(\lambda; H_D^{(L)})}{(2L)^d}.$$

Here, $H_D^{(L)}$ is the restriction of H to the cube $[-L, L]^d$ with the Dirichlet boundary conditions, and $N(\lambda; A) = \#\{\lambda_j(A) \le \lambda\}$ is the counting function of the discrete spectrum of A.

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If $d = 1$, then the number of gaps is almost always infinite.

Bethe-Sommerfeld conjecture

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If $d \geq 2$, the number of gaps of periodic Schrödinger operator $H = -\Delta + V$ is always finite.

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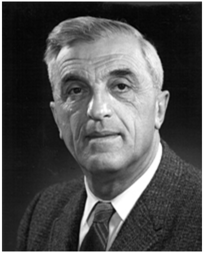
For the magnetic Schrödinger operator $H = (i\nabla + a)^2 + V$ the Bethe-Sommerfeld conjecture was proved only for $d = 2$ (A. Mohamed, 1997).

Important tool when working with periodic problems: Floquet-Bloch(-Brillouin-Gelfand) decomposition.



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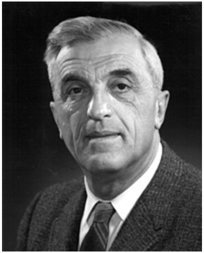

$$H = \int_{\oplus} H(k)dk,$$

where $H(k) = h(x, \xi + k)$ ($H(k) := (i\nabla + k)^2 + V$ in the Schrödinger case) acts in $L^2(\mathbb{R}^d/\Gamma)$, $k \in \mathbb{R}^d/\Gamma'$, and Γ' is the (analytical) dual to Γ (say $\Gamma' = \mathbb{Z}^d$). This means that

$$\sigma(H) = \cup_{k \in \mathbb{R}^d/\Gamma'} \sigma(H(k)).$$

The spectrum of $H(k)$ consists of eigenvalues:

$$\sigma(H(k)) = \{\lambda_1(k) \leq \lambda_2(k) \leq \dots\}.$$

Now we can define

$$\ell_j := \cup_{k \in \mathbb{R}^d/\Gamma'} \lambda_j(k)$$

as the n-th spectral band, so that $\sigma(H) = \cup_j \ell_j$. Then for each λ we can define two functions:

$$m(\lambda) = \#\{j : \lambda \in \ell_j\}$$

(the multiplicity of overlapping) and

$$\zeta(\lambda) = \zeta(\lambda; H) = \max_j \max\{t : [\lambda - t, \lambda + t] \subset \ell_j\}.$$

(the overlapping function)

Let us denote $N(\lambda; H(\mathbf{k})) = \#\{\lambda_j(\mathbf{k}) \leq \lambda\}$
Then we can express the density of states as

$$N(\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d/\Gamma^\dagger} N(\lambda, H(\mathbf{k})) d\mathbf{k}.$$

Theorem. (A.Sobolev,LP, 2001)

Let $d = 2, 3, 4$. Then for sufficiently large λ we have:

Dimension	$m(\lambda) \gg$	$\zeta(\lambda) \gg$
2	$\lambda^{\frac{1}{4}}$	$\lambda^{\frac{1}{4}}$
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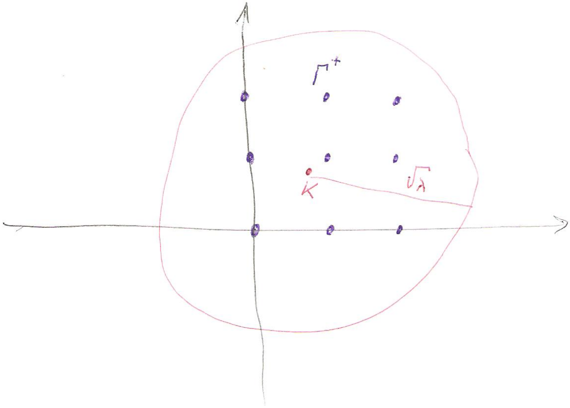
Unfortunately, the method does not work for $d \geq 5$!

The important part of the proof consists in the careful study of the unperturbed operator $H_0 = \Delta$. Then $N(\lambda; H_0(\mathbf{k}))$ is the number of integer points $\mathbf{n} \in \mathbb{Z}^d$ inside the ball of radius $\sqrt{\lambda}$ with centre at $\mathbf{k}$.



Γ^+
K
$\sqrt{\lambda}$

The integrated density of states $N(\lambda; H_0)$ is the average value of $N(\lambda; H_0(\mathbf{k}))$ averaged over all $\mathbf{k} \in \mathbb{R}^d/\Gamma'$; an easy calculation shows that we have

$$N(\lambda; H_0) = C_d \lambda^{d/2},$$

where

$$C_d = \frac{1}{2^d \pi^{d/2} \Gamma(1 + d/2)}.$$

Denote

$$S_1(\lambda) := \int_{\mathbb{R}^d/\Gamma'} |N(\lambda; H_0(\mathbf{k})) - N(\lambda; H_0)| d\mathbf{k}.$$

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$d \neq 1 (mod\ 4)$

If we want to prove the conjecture for all d and all lattices, we need to study the eigenvalues of $H(k)$. There are two types of eigenvalues of these operators: stable (corresponding to perturbations of simple eigenvalues, lying not too close to other eigenvalues) and unstable (corresponding to perturbations of a cluster of eigenvalues lying close together). It is relatively straightforward to compute stable eigenvalues with high precision. Unstable eigenvalues cause the main problem.

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Theorem. (G.Barbatis,LP, 2009)

Let $d \geq 2$. *Then the Bethe-Sommerfeld conjecture holds for operators* $H = (-\Delta)^m + q$ *with periodic pseudo-differential operators* q *of order smaller than* $2m - 1$.

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Let $d \geq 2$. Then the Bethe-Sommerfeld conjecture holds for operators $H = (-\Delta)^m + q$ with periodic pseudo-differential operators q of order smaller than $2m$. In particular, this conjecture holds for periodic magnetic Schrödinger operators.

Now let us move to the spectral questions of type (iii): the asymptotic behaviour of the integrated density of states. Recall: for positive λ we have

$$N(\lambda; H_0) = C_d \lambda^{d/2}.$$

There is a long-standing conjecture that the density of states of H enjoys the following asymptotic behaviour as $\lambda \to \infty$:

$$N(\lambda) \sim \lambda^{d/2}\Big(C_d + \sum_{j=1}^{\infty} e_j \lambda^{-j}\Big), \tag{1}$$

meaning that for each $K \in \mathbb{N}$ one has

$$N(\lambda) = \lambda^{d/2}\Big(C_d + \sum_{j=1}^{K} e_j \lambda^{-j}\Big) + R_K(\lambda) \tag{2}$$

with $R_K(\lambda) = o(\lambda^{\frac{d}{2}-K})$.

The coefficients e_j are real numbers which depend on the potential b. They can be calculated using the heat kernel invariants, computed by Polterovich, Hitrik-Polterovich, and Korotyaev-Pushnitski; they are equal to a certain integrals of the potential b and its derivatives. For example,

$$e_1 = -\frac{dw_d}{2(2\pi)^d|\mathbb{R}^d/\Gamma|}\int_{\mathbb{R}^d/\Gamma} b(\mathbf{x})d\mathbf{x}$$

and

$$e_2 = \frac{d(d-2)w_d}{8(2\pi)^d|\mathbb{R}^d/\Gamma|}\int_{\mathbb{R}^d/\Gamma}(b(\mathbf{x})^2)d\mathbf{x}.$$

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Yu.Karpeshina (2000) has shown that formula (2) is valid with $K = 1$ (two terms) and $R(\lambda) = O(\lambda^{-\frac{1}{105}})$ when $d = 3$ and $R(\lambda) = O(\lambda^{\frac{d-3}{2}} \ln \lambda)$ when $d > 3$.

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Theorem. (R.Shterenberg,LP, 2008–2012)

Formula (1) *holds in all dimensions.*

Almost-periodic problems

Now we want to study the density of states of quasi-periodic operators. First, we need to impose additional condition: let

$$V(\mathbf{x}) = \sum_{\theta \in \Theta} a_\theta e^{i\theta \mathbf{x}},$$

and let $Z(\Theta)$ be the collection of all linear combination of elements from Θ with integer coefficients. Let $\theta_1, \ldots, \theta_d \in Z(\Theta)$. Then either $\{\theta_j\}$ are linearly independent, or $\sum_{j=1}^d n_j \theta_j = 0$, where $n_j \in \mathbb{Z}$ and not all n_j are zeros.

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Theorem. (S.Morozov,R.Shterenberg,LP, 2012)

Let H be a magnetic Schrödinger operator where both the magnetic potential a and the electric potential V are quasi-periodic functions satisfying the above condition. Then formula (1) *holds.*

What is the analogue of the formula

$$N(\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d/\Gamma^\dagger} N(\lambda, H(\mathbf{k}))d\mathbf{k}$$

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Definition 1:
In all points of continuity of N, we have:

$$N(\lambda) = \mathbf{M}_{\mathbf{x}}(e(\lambda; \mathbf{x}, \mathbf{x})),$$

where $e(\lambda; \mathbf{x}, \mathbf{y})$ is the integral kernel of the spectral projection of H.

Definition 2 (cheating)

$$N(\lambda) = \mathbf{T}(E_\lambda(\tilde{H})) = \mathbf{D}(E_\lambda(\tilde{H})L^2(\mathbb{R}^d)).$$

Here, **T** is the regularized (von Neumann) trace, and **D** is the relative dimension.

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In particular, $N(\lambda; H) = N(\lambda; U^{-1}HU)$, where U is a unitary operator with almost-periodic coefficients.

Another useful trick: often, we work with operators acting not in $L_2(\mathbb{R}^d)$, but in $B_2(R^d)$ (Besicovitch space). This is a collection of all formal sums

$$\sum_j a_j e^{i\theta_j \mathbf{x}}$$

with

$$\sum_j |a_j|^2 < +\infty.$$

This is a non-separable Hilbert space. Results of Shubin show that the norms and spectra of almost-periodic operators acting in $L_2(\mathbb{R}^d)$ and $B_2(R^d)$ are often the same.